Two-Dimensional Heat Conduction Analysis Using Finite Element Method

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Abstract

Finite Element Method can be used to solve two-dimensional heat conduction problems. In this work, the internal heat generation was defined node-wise and a non-zero prescription of boundary normal heat flux was allowed. Using the Method of Manufactured Solutions, an example with known analytical solution was utilised to demonstrate the validity of the modified numerical method. The convergence plots of the temperature field using linear and quadratic triangular elements were obtained and compared, which agree well with theoretical convergence rates.

Introduction

The three modes of heat transfer are conduction, convection and radiation. Heat conduction is a process by which heat energy is transferred due to the physical contact of two bodies at different temperatures. Heat conduction is governed by a partial differential equation, which can be solved using finite element analysis (FEA).

In this work, an existing linear finite element code for 2D heat conduction is extended to allow the internal heat generation to be defined node by node and allow a non-zero prescription of boundary normal heat flux. A physical problem is solved using the extended code and the credibility of the results are discussed. A problem with an analytical solution is then defined to allow the variation in the accuracy of the finite element results with the mesh size to be assessed, using scripts which automatically generate mesh files and a function to compute the error. Finally, the 2D heat conduction finite element code is extended to allow quadratic triangular elements to be used. Using the same problem, the variation in the error obtained with the size of the mesh is discussed and compared with the results seen for linear elements.

Methodology

Problem Statement

The strong form of the steady state 2D heat flow problem is given as

$$\frac{\partial}{\partial x}\left[k(x,y)\frac{\partial T(x,y)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y)\frac{\partial T(x,y)}{\partial y}\right] + Q(x,y) = 0 \qquad (1)$$

where k is the thermal conductivity of the material, T is the temperature and Q is the heat source. Assuming isotropic conductivity, the weak form of Equation 1 can be invoked as

$$\int_{\Omega} \phi \Big[\frac{\partial}{\partial x}\Big(k\frac{\partial T(x,y)}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(k\frac{\partial T(x,y)}{\partial y}\Big)\Big] d\Omega + \int_{\Omega} \phi Q d\Omega = 0 \tag{2}$$

where Φ is a test function. Through applying the divergence theorem and rearranging, the weak form of the steady 2D heat flow problem is obtained as:

$$-\int_{\Gamma_q} \phi q_0 d\Gamma - \int_{\Gamma_c} \phi h(T - T_\infty) d\Gamma - \int_{\Omega} \Big(k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\Big) d\Omega + \int_{\Omega} \phi Q d\Omega = 0 \tag{3}$$

which gives rise to three different types of boundary conditions, one of which must be specified on every boundary of the domain. The three possible boundary conditions are:

- Prescribed boundary temperature $T = \bar{T}(x,y)$ for $x, y \in \Gamma_T$ (Dirichlet BC)
- Prescribed normal heat flux $q_n = \bar{q}_n(x,y)$ for $x, y \in \Gamma_q$ (Neumann BC)
- Convective boundary condition $q_n = h(T - T_\infty)$ for $x, y \in \Gamma_c$ (Cauchy BC)

where q_n is the flux normal to the boundary, h is the coefficient of convective heat transfer and T_∞ is the temperature of the convective medium.

An existing code for 2D steady heat transfer was modified to include new functionalities, where the internal heat generation is defined node-by-node.

The code was also modified to solve problems with non-zero prescription of boundary normal heat flux.

Example

The Method of Manufactured Solutions (MMS) [1] allows the required problem data (domain, heat source, conductivity and boundary conditions) for a given solution to be derived. The method begins with the governing partial differential equation of the problem, which in the case of heat transfer with constant conductivity is

$$k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + Q(x,y) = 0 \tag{4}$$

Equation 4 can be rearranged to give:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q(x,y)}{k} \tag{5}$$

The temperature field can be taken as any arbitrary function. The solution of the temperature field $T(x,y)$ was taken as

$$T(x,y) = 100 + \frac{1}{2}x^3 - 5y \tag{6}$$

The double derivatives in the governing partial differential equation were computed as:

$$\frac{\partial^2 T}{\partial x^2} = 3x \text{ and } \frac{\partial^2 T}{\partial y^2} = 0. \tag{7}$$

By substituting Equation 7 into Equation 5 and setting k = 1.0 W/m°C

$$Q(x, y) = -\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = -3x \tag{8}$$

The domain is rectangular and defined in a Cartesian coordinate system with the origin in the bottom left hand corner of the domain. Given that the width in the x-direction is W and the length in the y-direction is H, the boundary conditions for all four sides of the domain can be derived as:

$$\begin{aligned} T(0, y) &= 100 - 5y \qquad & T(W, y) &= 100 + \frac{1}{2}W^3 - 5y \\ T(x, 0) &= 100 + \frac{1}{2}x^3 \qquad & T(x, H) &= 100 + \frac{1}{2}x^3 - 5H \end{aligned} \tag{9}$$

A diagram of the problem and the boundary conditions is shown in Figure 1.

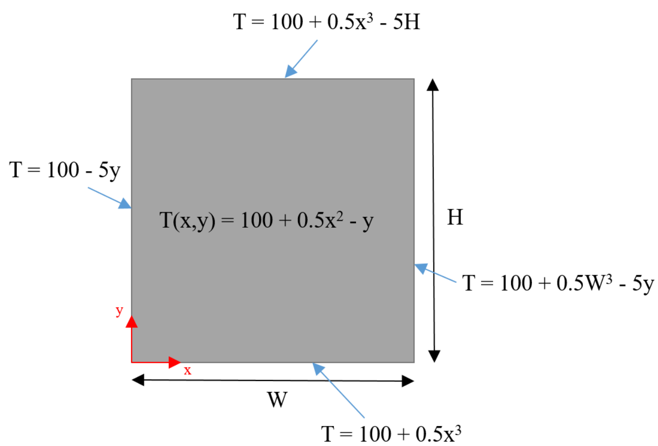


Figure 1: Problem with applied boundary conditions

Results

An L^2-norm error metric was used to quantify the error between the analytical solution (from Equation 6) and the finite element solution.

$$L^2(U) = ||U||_{L^2(\Omega)} = \sqrt{\left(\int_\Omega U^2 d\Omega \right)} \tag{10}$$

The error e in a finite element solution is defined as the difference between the exact solution u and the approximate finite element solution $\bar{u}$ ($e = u - \bar{u}$). Hence the L^2-norm of the error in the finite element solution is defined as

$$L^2(e) = ||e||_{L^2(\Omega)} = \sqrt{\Big(\int_\Omega e^2 d\Omega\Big)} = \sqrt{\Big(\int_\Omega (u-\bar{u})^2 d\Omega\Big)} \tag{11}$$

The error metric used to quantify the error in this work was defined as a relative error

$$E_r = \frac{||e||_{L^2(\Omega)}}{||u||_{L^2(\Omega)}} \tag{12}$$

The finite element solution was obtained for each mesh where the height H and width W of the plate were both set as 10. The error in a finite element solution depends on the mesh size h and the order of elements used p. The relationship is given as [2]

$$||e||_{L^2(\Omega)} \approx Ch^{(p+1)} \tag{13}$$

where C is a constant. Taking $||e||_{L^2(\Omega)}$ to be the relative error E_r defined in Section **??** and taking the logarithm of both sides of Equation 13, then

$$log(E_r) = (p+1)\log(h) + log(C) \tag{14}$$

meaning that the gradient of $\log(E_r)$ plotted against $\log(h)$ should be approximately equal to $p+1$. Figure 2 shows the variation in the error E_r with the mesh element size h.

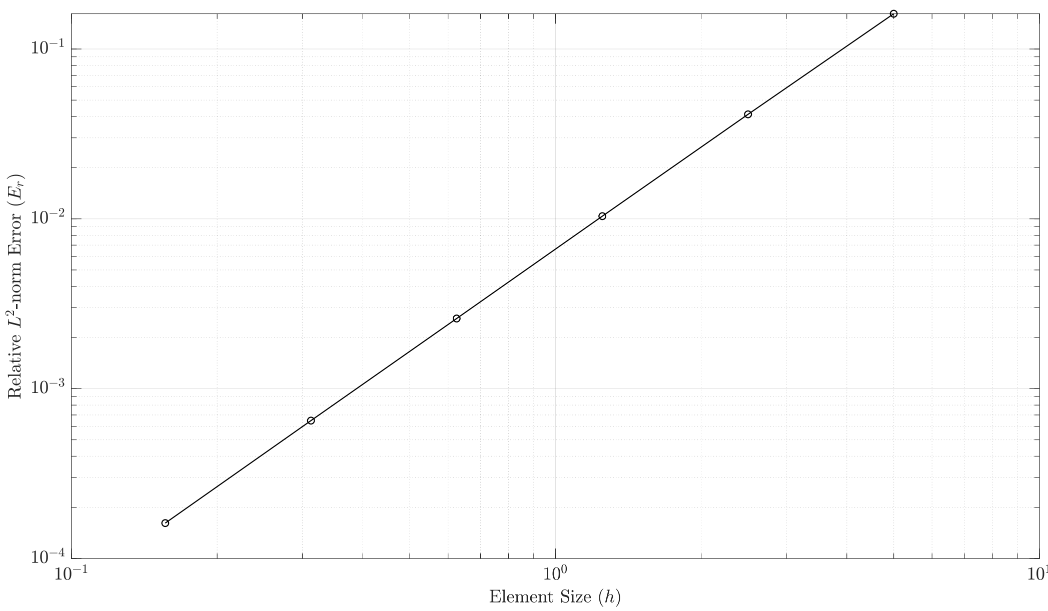
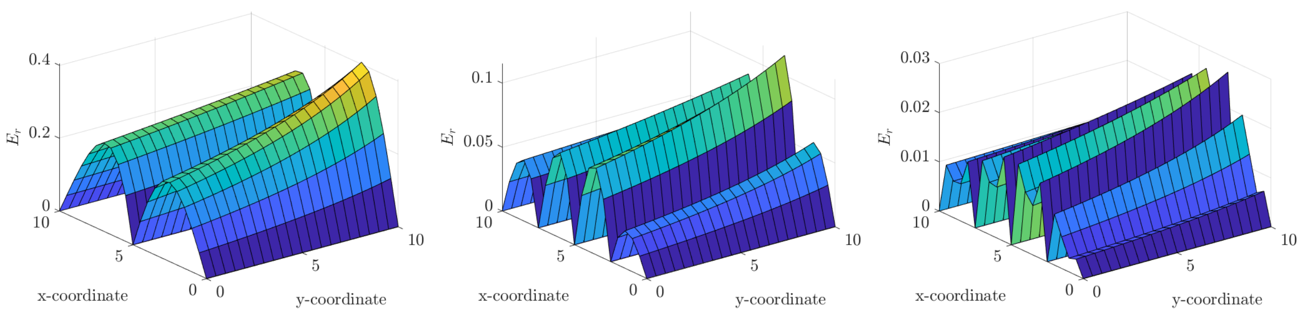


Figure 2: Variation in the error with the mesh element size

As linear elements were used to perform the finite element analyses, $p = 1$. The gradient of the line in Figure 2 is 1.993 (to 3 decimal places), which is consistent with Equation 13. This confirms that the finite element code is correct. From this it can be concluded that reducing the mesh size by a factor of k gives a reduction in the error by a factor of $k^{(p+1)}$. Figure 3 shows the variation in the error throughout the domain for the first three meshes.



(a) Mesh 1 (b) Mesh 2 (c) Mesh 3

Figure 3: Surface plots of the relative L^2-norm error E_r for the first three meshes

The linear elements were able to accurately capture the linear boundary condition at $x = 0$ and $x = W = 10$. The temperature on the boundaries at $y = 0$ and $y = H = 10$ was defined by a cubic function, which the linear elements struggled to capture. This can be seen from the error peaking in the middle of the elements as there were 2 elements in the y-direction in mesh 1, 4 elements in the y-direction in mesh 2, and 8 elements in the y-direction in mesh 3.

Figure 4 shows the variation in the error E_r with the mesh element size h when quadratic elements were used. The equivalent plot for linear elements (Figure 2) is also shown in red for reference.

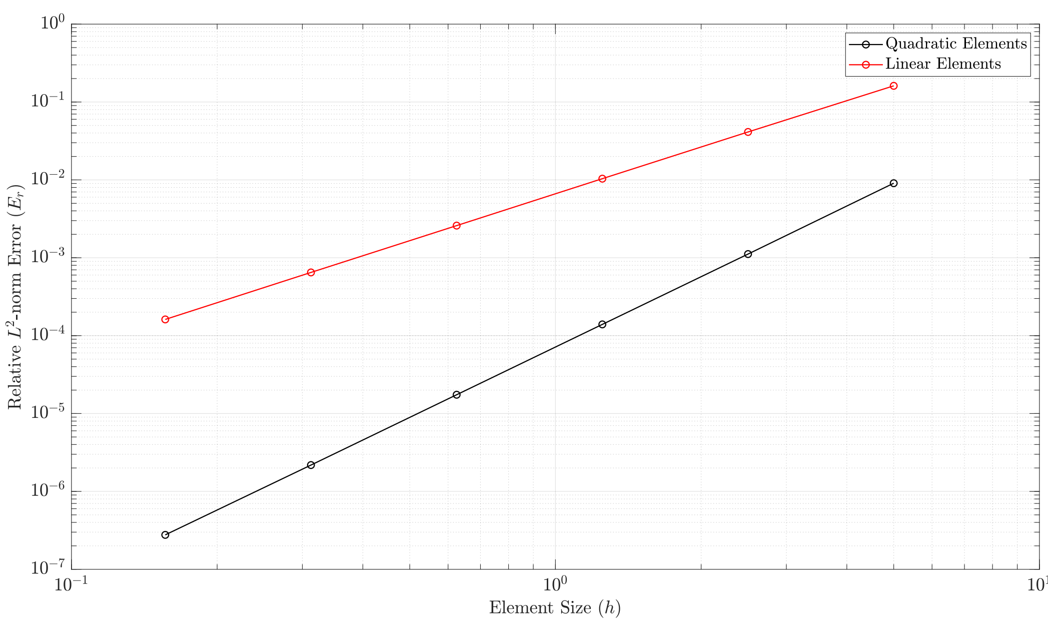


Figure 4: Variation in the error with the mesh element size

Theoretically, it was expected that the gradient of the line would be $p + 1$, where p is 2 in the case of quadratic elements. The gradient of the line in Figure 4 was 3.000 (to 3 decimal places), confirming that the extension to the code produces correct results. Reducing the mesh size of quadratic elements by a factor of k gave reduction in error of k^3, whereas reducing the mesh size of linear elements by the same factor gave a reduction in error of k^2. In other words, the rate of convergence was higher when using quadratic elements. Using quadratic elements also gave a lower error then using linear elements when the same number of elements were used. For example when a mesh size of 5 was used, the error was 16.1% when using linear elements compared to just 0.91% when using quadratic elements.

Conclusions

An existing code for 2D steady heat transfer was successfully modified to allow problems with internal heat generation defined node-by-node and a non-zero prescription of boundary normal heat flux to be solved. The mathematical derivations of the heat source term and boundary term in the weak form of the 2D steady state heat conduction problem were presented. The modified code was tested by solving a physical problem, which validated that the FE solution was realistic.

The modified code was then used to solve a physical problem with a known exact solution. This problem was used to investigate the relationship of the error between the approximate FE solutions and the known exact solution with the element size. The accuracy of the FE solutions were validated.

The code was then extended to allow the use of quadratic triangular elements and the error convergence validated. It was found that using quadratic elements gives a lower error than using linear elements, and the rate of convergence when using quadratic elements is higher.

References

[1] K. Salari and P. Knupp. *Code Verification by the Method of Manufactured Solutions.* Sandia Report, Sandia National Laboratories, 2000.

[2] J. E. Akin. *Finite Element Analysis with Error Estimators.* Elsevier Science Technology, 2005.