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H.W-3

2. a)

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_{i+1}^{n+1} - U_i^{n+1}}{\Delta x} = 0$$

$$U_i^{n+1} - U_i^n + r\left(U_{i+1}^{n+1} - U_i^{n+1}\right) = 0 \qquad \text{with } r = \frac{a\Delta t}{\Delta x}$$

$$U_i^n = (1-r)U_i^{n+1} + rU_{i+1}^{n+1}$$

b)

$$U_0^n = (1-r)\,U_0^{n+1} + rU_1^{n+1}$$

$$U_1^n = (1-r)\,U_1^{n+1} + rU_2^{n+1}$$

$$\vdots \qquad \qquad \vdots$$

$$U_{m-1}^n = (1-r)U_{m-1}^{n+1} + rU_m^{n+1}$$

Periodic boundary conditions are treated as linear equation.

c) Direct method → Doolittle

Iterative → Gauss-Siedel

d) Fill in matrix:

$$\begin{bmatrix} (1-r) & r & 0 & \cdots & & \\ 0 & (1-r) & r & \cdots & & \\ 0 & 0 & (1-r) & r & \cdots & \\ & & & \ddots & \ddots & \\ & & & & & r \\ & & & & 1-r & r \\ r & \cdots & & & & 1-r \end{bmatrix}$$

4) a) $U_t = \nu U_{xx} + \sigma U$

$$\Rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t} = \nu\left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}\right) + \sigma U_i^n$$

$$\Rightarrow U_i^{n+1} = \frac{\nu \Delta t}{\Delta x^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \Delta t \sigma U_i^n + U_i^n$$

$$\Rightarrow U_i^{n+1} = r\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \Delta t \sigma U_i^n + U_i^n \quad \left(r = \frac{\nu \Delta t}{\Delta x^2}\right)$$

$$\Rightarrow U_i^{n+1} = rU_{i+1}^n + (1 + \Delta t\sigma - 2r)U_i^n + rU_{i-1}^n$$

incorporating initial and boundary conditions:

$U(0,t) = 0$ [Direchlet]

$U_x(1,t) = 0 \rightarrow$ newman,

we use, fictitious node to deal with neuman boundary condition.

$\cdots$

Using central difference at $i = m$

$$\frac{du}{dx} = \frac{U_{m+1} - U_{m-1}}{2\Delta x} = 0$$

$$\therefore U_{m+1} = U_{m-1}$$

$$U_m^{n+1} = rU_{m+1}^n + (1 + \Delta t\sigma - 2r)U_m^n + U_{m-1}^n$$

b) $\sigma = 0$,

$$U_i^{n+1} = rU_{i+1}^{n} + (1-2r)U_i^{n} + U_{i-1}^{n}$$

$v = 0$

$$U_i^{n+1} = \Delta t \sigma U_i^{n} + U_i^{n}$$

c) Using, $v = 0.1$, $\sigma = -0.1$, $\Delta x = 0.25$, $\Delta t = 0.1$,

$$r = \frac{v \Delta t}{(\Delta x)^2} = \frac{0.1 \times 0.1}{(0.25)^2} = 0.16$$

$U_0^0 = 0$, $U_1^0 = 0$, $U_2^0 = 1$, $U_3^0 = 0$, $U_4^0 = 0$

1st time step ($n=0$)

$U_0^1 = 0$

$$U_1^1 = rU_2^0 + (1 + \sigma\Delta t - 2r)U_1^0 + rU_0^0$$

putting the values of r, U_2^0, σ, Δt, U_1^0, U_0^0

we get, $U_1^1 = 0.16$

in a similar way, other values are found,

$$U_2^1 = rU_3^0 + (1 + \sigma\Delta t - 2r)U_2^0 + rU_1^0$$
$$= 0.67$$

$$U_3^1 = rU_4^0 + (1 + \sigma\Delta t - 2r)U_3^0 + rU_2^0$$
$$= 0.16$$

$$U_4^1 = rU_5^0 + (1 + \sigma\Delta t - 2r)U_4^0 + rU_3^0 \quad [U_5^0 = U_3^0]$$
$$= 0$$

2nd time step (n = 1)

$U_0^2 = 0$

$U_1^2 = rU_2^1 + (1 + \sigma\Delta t - 2r)U_1^1 + rU_0^1 = 0.21$

$U_2^2 = rU_3^1 + (1 + \sigma\Delta t - 2r)U_2^1 + rU_1^1 = 0.50$

$U_3^2 = rU_4^1 + (1 + \sigma\Delta t - 2r)U_3^1 + rU_2^1 = 0.21$

$U_4^2 = rU_5^1 + (1 + \sigma\Delta t - 2r)U_4^1 + rU_3^1 = 0.05$

d) $$\frac{U_i^{n+1} - U_i^n}{\Delta t} = v\left(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2}\right) + \sigma U_i^{n+1}$$

$$U_i^{n+1} - U_i^n = \frac{v\Delta t}{\Delta x^2}\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right) + \sigma\Delta t\, U_i^{n+1}$$

$$U_i^n = -rU_{i+1}^{n+1} + (1 + 2r - \sigma\Delta t)U_i^{n+1} - rU_{i-1}^{n+1}$$

initial condition

$U(0,t) = 0$ (directlet)

$U_x(1,t) = 0$ (neuman)

$\downarrow$

we used fictitous node theory
and simplify above general equation ~~by~~
~~replacing~~

The matrix

$$\begin{bmatrix} 1+2r-\sigma\Delta t & -r & & & \\ -r & 1+2r-\sigma\Delta t & -r & & \\ & -r & 1+2r-\sigma\Delta t & -r & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & -r \\ & & & -r & 1+2r-\sigma\Delta t \end{bmatrix}$$

We can use ~~detom~~ Gauseidel method to solve this.