EXERCISES 1

(1) $f(x) = x^3 + 2x^2 + 10x - 20 = 0$

4 iterations, Newton's method, $x^0 = \sqrt[3]{20}$

$$\left.\begin{aligned} x^{k+1} &= x^k + \Delta x^{k+1} \\ \Delta x^{k+1} &= -\frac{f(x^k)}{f'(x^k)} \end{aligned}\right\} \text{Newton's algorithm}$$

For this equation, $f'(x) = 3x^2 + 4x + 10$

ITERATION 1 $\quad x^1 = x^0 + \Delta x^1$

$$\Delta x^1 = -\frac{(x^0)^3 + 2(x^0)^2 + 10x^0 - 20}{3(x^0)^2 + 4(x^0) + 10} = -0'974825$$

$$x^1 = \sqrt[3]{20} - 0'974825 = 1'73959$$

ITERATION 2

$$\Delta x^2 = -\frac{(x^1)^3 + 2(x^1)^2 + 10x^1 - 20}{3(x^1)^2 + 4(x^1) + 10} = -0'334625$$

$$x^2 = x^1 + \Delta x^2 = 1'73959 - 0'334625 = 1'40497$$

ITERATION 3

$$\Delta x^3 = -\frac{(x^2)^3 + 2(x^2)^2 + 10(x^2) - 20}{3(x^2)^2 + 4(x^2) + 10} = -0'035784$$

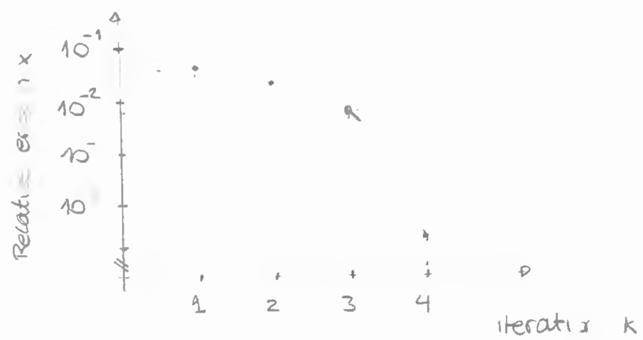
$$x^3 = x^2 + \Delta x^3 = 1'40497 - 0'035784 = 1'36918$$

ITERATION 4

$$\Delta x^4 = -\frac{(x^3)^3 + 2(x^3)^2 + 10(x^3) - 20}{3(x^3)^2 + 4(x^3) + 10} = -0'000374979$$

$$x^4 = x^3 + \Delta x^4 = 1'36918 - 0'000374979 = 1'36881$$

CONVERGENCE IN X



$$e^1 = \frac{x_0 - x_1}{x_1} = 0'560 = 5'6 \; 10^{-1}$$

$$e^2 = \frac{x_1 - x_2}{x_2} = 0'2381 = 2'3 \; 10^{-1}$$

$$e^3 = \frac{x_2 - x_3}{x_3} = 0'026 = 2'6 \; 10^{-2}$$

$$e^4 = \frac{x_3 - x_4}{x_4} = 0'0003 = [illegible] \; 10^{-4}$$

The method behaves as expected, with the relative error reducing quadratically with the number of iterations.

(5) Definition of a 3^{rd} order quadrature in (0,1)

a) Minimum number of integration points and specify integration points and weights.

3^{rd} order quadrature $\rightarrow$ integrates exactly polynomials of degree ≤ 3

Minimum number of integration points $\rightarrow$ 2, (using Gaussian quad.)

For Gauss Legendre quad and $n = 1$: $z_0 = -\sqrt{3}/3$, $z_1 = \sqrt{3}/3$, $w_0 = w_1 = 1$

$$\int_{-1}^{1} f(z)\,dz \simeq \sum_{i=0}^{n} w_i f(z_i)$$

The integral is defined from 0 to 1 so a change of variables is needed:

$$x = \frac{b-a}{2} z + \frac{a+b}{2} \qquad dx = \frac{b-a}{2} dz$$

$$I = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} z + \frac{a+b}{2}\right) dz = \frac{b-a}{2} \int_{-1}^{1} f(z)\,dz$$

Using Gauss-Legendre: $I \simeq \frac{b-a}{2} \sum_{0}^{n} w_i f(z_i)$

In this problem $a = 0$, $b = 1$

$$x_0 = \frac{1}{2}\left(-\frac{\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3-\sqrt{3}}{6} \qquad w_0 = 1$$

$$x_1 = \frac{1}{2}\frac{\sqrt{3}}{3} + \frac{1}{2} = \frac{3+\sqrt{3}}{6} \qquad w_1 = 1$$

b) To integrate using those points it will be necessary to have a method that work for combined intervals (one end closed and one end open)

⑥ a) If $n+1$ points Gaussian quadrature is used, state the order of the polynomial that is exactly integrated.

With $n+1$ points, polynomials of degree $2n+1$ can be integrated using Gaussian quadrature

b) if $n=2$, which will be exactly integrated?

if $n=2$, $2n+1 = 5$ order polynomials can be integrated

i) $\int_0^1 \sin x \, dx$ NO

ii) $\int_0^1 x^3 dx$ YES

iii) $\int_0^1 x^4 dx$ YES

iv) $\int_0^1 x^{5.5} dx$ NO

⑦ $\underbrace{\int_0^1 (\overbrace{12x}^{f_1}) dx}_{I_1}$ & $\underbrace{\int_0^1 (\overbrace{5x^3 + 2x}^{f_2}) dx}_{I_2}$ using

a) Trapezoidal rule with 2 intervals

	$x_0 = 0$	$x_1 = 0'5$	$x_2 = 1$
f_1	0	6	12
f_2	0	1'625	7

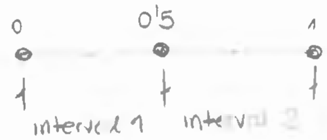
$I_1 = \frac{0'5}{2}\left(f_1(x_0) + f_1(x_1)\right) + \frac{0'5}{2}\left(f_1(x_1) + f_1(x_2)\right) = 6$

$I_2 = \frac{0'5}{2}\left(f_2(x_0) + 2 f_2(x_1) + f_2(x_2)\right) = 2'5625$

$e_1 = |\text{exact value} - I_1| = 0$ (as expected because it's a first-order polynomial that can be exactly integrated with trapezoids)

$e_2 = |\text{exact value} - I_2| = 0'138$ (as expected it can't be exactly integrated)

b) Simpson's rule with 2 intervals



	x_0	x_1	x_2	x_3	x_4
f_1	0	3	6	9	12
f_2	0	0'8125	2'25	4'3125	7

$$I_1 = \frac{0'25}{3}\left(f_1(x_0) + 4f_1(x_1) + f_1(x_2)\right) + \frac{0'25}{3}\left(f_1(x_2) + 4f_1(x_3) + f_1(x_4)\right) = 6$$

$$I_2 = \frac{0'25}{3}\left(f_2(x_0) + 4f_2(x_1) + 2f_2(x_2) + 4f_2(x_3) + f_2(x_4)\right) = 2'6667$$

$e_1 = 0$ } as expected because both can be exactly integrated

$e_2 = 0$ } using Simpson rule

(10) $$\int_0^1\int_0^1 \overbrace{(9x^3+8x^2)}^{f(x)}\,\overbrace{(y^3+y)}^{g(y)}\,dx\,dy$$

$h = 0'25$

	x_0	x_1	x_2	x_3	x_4
f	0	0'640625	3'125	8'29688	17
g	0	0'265625	0'625	1'17188	2

$$I_f = \frac{0'25}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) = 4'91667$$

$$I_g = \frac{0'25}{3}\left(g(y_0) + 4g(y_1) + 2g(y_2) + 4g(y_3) + g(y_4)\right) = 0'75$$

$$I = I_f\, I_g = 3'6875$$

$e = |\text{exact value} - I| = 0$ → as expected, because both polynomials were independent and of order 3, so they could be exactly integrated with Simpson's rule