Question 1/ $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$

$t = 1$

Initial condition: $\theta(1) = 0{,}4$ rad; $\frac{d\theta(1)}{dt} = 0$ rad/s

Using the second order Runge-Kutta:

$$\bar\theta = \begin{bmatrix} \theta \\ \frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \bar\theta' = f(t,\theta) = \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ -\frac{g}{L}\theta_1 \end{bmatrix}$$

Remarks: The second order Runge-Kutta is explicit method:

For 2 time step $h = 0{,}5$, For 4 time step $h = 0{,}25$

1) 2 step.: To solve we write following formula:

1st step $y^*_{i+1} = y_i - hf(x_i, y_i)$; $\theta_1 = 0{,}4$, $\theta_2 = 0$, $t = 0{,}5$

$$y_{i+1} = y_i - \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}; y^*_{i+1})$$

$$\theta^*_{i+1} = \theta_i - hf(t_i, \theta_i)$$

$$\theta_{i+1} = \theta_i - \frac{h}{2}[f(t_i, \theta_i) + f(t_{i+1}, \theta^*_{i+1})]$$

for $i = 0$, $\theta^*_1 = \theta_0 - hf(t_0, \theta_0) = \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -9{,}8 \times 0{,}4 \end{bmatrix} =$

$$= \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1{,}96 \end{bmatrix} = \begin{bmatrix} 0{,}4 \\ 1{,}96 \end{bmatrix}$$

$$\theta_1 = \theta_0 - \frac{h}{2}[f(t_0, \theta_0) + f(t_1, \theta^*_1)] =$$

$$= \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ -9{,}8 \cdot 0{,}4 \end{bmatrix} + \begin{bmatrix} 1{,}96 \\ -9{,}8 \cdot 0{,}4 \end{bmatrix}\right) = \begin{bmatrix} -0{,}09 \\ 1{,}96 \end{bmatrix}$$

2nd step.; $t = 0.$ $\theta'_1 = -0{,}09$, $\theta'_2 = 1{,}96$

for $i = 1$; $\theta^*_2 = \theta_1 - hf(t_1, \theta_1) = \begin{bmatrix} -0{,}09 \\ 1{,}96 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1{,}96 \\ -9{,}8(-0{,}09) \end{bmatrix}$

$$= \begin{bmatrix} -0{,}09 \\ 1{,}96 \end{bmatrix} - \begin{bmatrix} 0{,}98 \\ 0{,}441 \end{bmatrix} = \begin{bmatrix} -1{,}07 \\ 1{,}519 \end{bmatrix}$$

$$\theta_2 = \theta_1 - \frac{1}{4}[f(t_1, \theta_1) + f(t_2, \theta^*_2)] =$$

$$= \begin{bmatrix} -0{,}09 \\ 1{,}96 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 1{,}96 \\ -9{,}8 \cdot (-0{,}09) \end{bmatrix} + \begin{bmatrix} 1{,}519 \\ -9{,}8 \cdot (-1{,}07) \end{bmatrix}\right) =$$

$$= \begin{bmatrix} -0{,}95975 \\ -0{,}882 \end{bmatrix}$$

(2) 4 time step. : $h = 0{,}25$, $t = 0{,}75$ san

for $i=0$

$$\theta_0 = \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix}$$

$$\theta_1^* = \theta_0 - hf(t_0, \theta_0) = \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 \\ -9{,}8 \cdot 0{,}4 \end{bmatrix} =$$

$$= \begin{bmatrix} 0{,}4 \\ 0{,}98 \end{bmatrix}$$

$$\theta_1 = \theta_0 - \frac{h}{2}\left[f(t_0, \theta_0) + f(t_1, \theta_1^*)\right] =$$

$$= \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 0 \\ -9{,}8 \cdot 0{,}4 \end{bmatrix} + \begin{bmatrix} 0{,}98 \\ -9{,}8 \cdot 0{,}4 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} 0{,}2775 \\ 0{,}98 \end{bmatrix}$$

for $i=1$, $t = 0{,}5$ san.

$$\theta_2^* = \theta_1 - hf(t_1, \theta_1) = \begin{bmatrix} 0{,}2775 \\ 0{,}98 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0{,}98 \\ -9{,}8 \cdot 0{,}2775 \end{bmatrix} =$$

$$= \begin{bmatrix} 0{,}0325 \\ 1{,}66 \end{bmatrix}$$

$$\theta_2 = \theta_1 - \frac{h}{2}\left[f(t_1, \theta_1) + f(t_2, \theta_2^*)\right] = \begin{bmatrix} 0{,}2775 \\ 0{,}98 \end{bmatrix} -$$

$$- \frac{1}{8}\left(\begin{bmatrix} 0{,}98 \\ -9{,}8 \cdot 0{,}2775 \end{bmatrix} + \begin{bmatrix} 1{,}66 \\ -9{,}8 \cdot 0{,}0325 \end{bmatrix}\right) = \begin{bmatrix} -0{,}0525 \\ 1{,}3598 \end{bmatrix}$$

for $i = 2$, $t = 0{,}25$ san.

$$\theta_3^* = \theta_2 - hf(t_2, \theta_2) = \begin{bmatrix} -0{,}0525 \\ 1{,}3598 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1{,}3598 \\ -9{,}8 \cdot (-0{,}0525) \end{bmatrix} =$$

$$= \begin{bmatrix} -0{,}3924 \\ 1{,}2312 \end{bmatrix}$$

$$\theta_3 = \theta_2 - \frac{h}{2}\left(f(t_2, \theta_2) + f(t_3, \theta_3^*)\right) = \begin{bmatrix} -0{,}0525 \\ 1{,}3598 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 1{,}3598 \\ 0{,}5145 \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} 1{,}2312 \\ -9{,}8 \cdot (-0{,}3924) \end{bmatrix}\right) = \begin{bmatrix} -0{,}3763 \\ 0{,}8147 \end{bmatrix}$$

for $i = 3$, $t = 0$ san.

$$\theta_4^* = \theta_3 - hf(t_3, \theta_3) = \begin{bmatrix} -0{,}3763 \\ 0{,}8147 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0{,}8147 \\ -9{,}8 \cdot (0{,}3763) \end{bmatrix} =$$

$$= \begin{bmatrix} -0{,}581 \\ -0{,}1073 \end{bmatrix}$$

$$\theta_4 = \bar{\theta}_3 - \frac{h}{2}\left[f(t_3,\bar{\theta}_3) + f(t_4, \theta_4^*)\right] =$$

$$= \begin{bmatrix} -0{,}3763 \\ 0{,}8147 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 0{,}8147 \\ -9{,}8\cdot(-0.3763) \end{bmatrix} + \begin{bmatrix} -0{,}1073 \\ -9{,}8\cdot(-0{,}581) \end{bmatrix}\right) =$$

$$= \begin{bmatrix} -0{,}4648 \\ -0{,}3568 \end{bmatrix}$$

As a result, for 2 steps at $t=0$ $\theta = -0{,}95975$,
for 4 time steps. $\theta = -0.464777 \simeq -0{,}4648$
Our analytical solution: $\theta(t) = 0{,}4\cos\left(\sqrt{\frac{g}{L}}(t-1)\right)$

b) Relative error: $t=0$. $RE = \left|\frac{\theta - \theta_e}{\theta}\right|$

$\theta_{2step} = -0{,}95975$ rad.

We get exact solution $\rightarrow \theta$ from analytical solution, so $\theta = -0{,}4$.

The relative error is computed as:

$$RE = \left|\frac{-0{,}4-(-0{,}95975)}{-0{,}4}\right| = \left|\frac{-0{,}4+0{,}95975}{-0{,}4}\right| = 1{,}399375$$

$$= 1{,}399375$$

for 4 time steps relative error as following,

$\theta_{4steps} = -0{,}464777$ rad

$$RE = \left|\frac{-0{,}4+0{,}464777}{-0{,}4}\right| = 0{,}1619425$$

c) $$\frac{E_{h^*}}{E_h} = \frac{C(h^*)^{p+1}}{Ch^{p+1}} \rightarrow h^* = \left(\frac{Eh^*}{Eh}\right)^{\frac{1}{p+1}} \cdot h$$ the order of method $\boxed{P=2}$

$h=0{,}25$, For RK2, in order to obtain relative error three orders of magnitude smaller:

$$h^* = (10^{-3})^{\frac{1}{2+1}} \cdot 0{,}25 = 0{,}025$$

(2) (a) $\frac{dy}{dx} = y - x^2 + 1$; $y(0) = 1$, $x \in (0,1)$

$f(x,y) = y - x^2 + 1$

$x_0 = 0, \quad y_0 = 1 \quad , h = 0,25$

$x_{n+1} = x_n + h$

$x_1 = x_0 + h = 0,25$

$y_{n+1} = y_n + h f(x_n, y_n)$, $f(x_0, y_0) = 2$

• Y_1 $\quad y_1 = y_0 + h f(x_0, y_0) = 1 + 0,25 \cdot 2 = 1,5$

$x_1 = 0,25 \quad ; \quad y_1 = 1.5;$

• Y_2 $\quad x_2 = x_1 + h = 0,25 + 0,25 = 0,5$

$y_2 = y_1 + h f(x_1, y_1) = 1,5 + 0,25 \cdot 2,4375 = 2,1094$

$f(x_1, y_1) = f(0,25; 1,5) = 1,5 - 0,25^2 + 1 = 2,4375$

$x_2 = 0,5 \quad ; \quad y_2 = 2,1094.$

• Y_3 . $\quad x_3 = x_2 + h = 0,5 + 0,25 = 0,75.$

$y_3 = y_2 + h f(x_2, y_2) = 2,1094 + 0,25 \cdot 2,8594 = 2,8243$

$f(x_2, y_2) = f(0,5; 2,1094) = 2,1094 - 0,5^2 + 1 = 2,8594$

• Y_4: $\quad x_4 = x_3 + h = 0,75 + 0,25 = 1$

$y_4 = y_3 + h f(x_3, y_3) = 2,8243 + 0,25 \cdot 3,2618$
$= 3,640.$

$f(x_3, y_3) = f(0,75; 2,8243) = 2,8243 - 0,75^2 + 1 =$
$= 3,2618$

$y_4 = Y_4 = 3,64.$

(b). We use Heun method; Heun Method is also second order Runge-Kutta method.

$$\begin{bmatrix} Y^{*}_{i+1} = Y_i + h f(x_i, Y_i) \\ Y_{i+1} = Y_i + \frac{h}{2}[f(x_i, Y_i) + f(x_{i+1}, Y^{*}_{i+1})] \end{bmatrix}$$ (explicit method)

$$\begin{cases} y_{i+1} = \frac{h}{2}\,[k_1+k_2] + y_i \\ k_1 = f(x_i, y_i)\ ; \quad k_2 = f(x_i+h,\ y_i+hk_1) \end{cases}$$

$h = 0,25,\ x_0 = 0,\ y_0 = 1\ ;\ f(x,y) = y - x^2 + 1$

$y_1 = \frac{h}{2}\,[k_1+k_2] + y_0$

$y_1 = y_0 + \frac{1}{8}\,[k_1+k_2]$

$k_1 = f(x_0, y_0) = 2.$

$k_2 = f(x_0+h,\ y_0+hk_1) = f(0,25,\ 1+0,25\cdot 2) =$

$= f(0,25,\ 1,5) = 1,5 - 0,25^2 + 1 = 2,4375.$

$y_1 = 1 + \frac{1}{8}\cdot[2 + 2,4375] = 1 + 0,5547 = \underline{1,5547}.$

$y_2 = y_1 + \frac{h}{2}\,[K_1+K_2] = 1,5547 + \frac{1}{8}\,[2,4922 + 2,9278] =$
$= \underline{2,2322}.$

$x_1 = x_0 + h = 0,25$

$K_1 = f(x_1, y_1) = f(0,25;\ 1,5547) = 1,5547 - 0,25^2 + 1 = 2,4922$

$K_2 = f(x_1+h,\ y_1+hK_1) = f(0,5;\ 1,5547 + 0,25\cdot 2,4922) =$

$= f(0,5;\ 2,1778) = 2,1778 - 0,5^2 + 1 = \underline{2,9278}.$

$y_3 = y_2 + \frac{h}{2}\,[K_1+K_2] = 2,2322 + \frac{1}{8}\,[2,9822 + 2,5403] =$

$= 2,2322 + 0,6903 = \underline{2,9225}.$

$x_2 = x_1 + h = 0,5.$

$K_1 = f(x_2, y_2) = f(0,5;\ 2,2322) = 2,2322 - 0,5^2 + 1 =$
$= 2,9822.$

$k_2 = f(x_2+h,\ y_2+hK_1) = f(0,75,\ 2,2322 + 0,25\cdot 2,9822) =$
$= f(0,75;\ 2,9778) = 2,9778 - 0,75^2 + 1) = 2,5403$

$x_3 = x_2 + h = 0,75.$

$y_4 = y_3 + \frac{h}{2}\,[K_1+K_2] = 2,9225 + \frac{1}{8}\,[3,36 + 3,7625] =$
$= \underline{3,8128}.$

$K_1 = f(x_3, y_3) = f(0,75;\ 2,9225) = 2,9225 - 0,75^2 + 1 = 3,36.$

$K_2 = f(x_3+h;\ y_3+hK_1) = f(\cancel{0,75}\ 1;\ 2,9225 + 0,25\cdot 3,36) =$

$f(\cancel{0,75}\ 1;\ 3,7625) = 3,7625$

Question 3: Solution:

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(0)=1 \text{ (Initial condition)}$$

$$Y_{i+1} = Y_i + h\, f(x_i, Y_i)$$

a) Taylor series: $y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \dots$ (1)

Equation verified by the analytical solution:

$$y_{i+1} = y_i + h\, f(x_i, y_i) + h\, \tau_i(h) \rightarrow \text{truncation error}$$

neglecting the truncation errors, the numerical scheme of the Euler method is obtained:

$$y_{i+1} = y_i + h\, f(x_i, y_i) \rightarrow$$ equation verified by the numerical solution;

$$h\tau_i(h) = R_i(h) = y_{i+1} - [y_i + h f(x_i, y_i)] =$$
$$= [\cancel{y_i} + h y_i' + y_i'' \frac{h^2}{2} + \dots] - \cancel{y_i} - h\, \underbrace{f(x_i, y_i)}_{y_i' \text{ (ODE)}}$$

$$h\, \tau_i(h) = \frac{h^2}{2} y_i'' + O(h^3) \rightarrow \tau_i(h) = \frac{h}{2} y_i'' + O(h^2).$$

Or equivalently; $\tau_i(h) = \frac{h}{2} y_i''(\xi_i)$

Remark: the truncation error can also be deduced from the derivation of the method

$$(1) \Rightarrow y_{i+1} = y_i + h\, f(x_i, y_i) + \underbrace{\frac{h^2}{2} y_i'' + \dots}_{h\tau_i(h) \leftarrow \text{neglected truncation error}}$$

$$Y_{i+1} = Y_i + h\, f(x_i, Y_i)$$

Yes, the method is said consistent, because

$\max_{0 \le i \le m} \tau_i(h) \rightarrow 0,\ h \rightarrow 0$, for any "well posed" initial value problem

b) Taylor series: $y_i = y_{i+1} - h \cdot y_{i+1}' + \frac{h^2}{2} y_{i+1}'' + O(h^3)$

replacing the ODE and rearranging terms:

$$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) + \tau_i(h)$$

with truncation error $\tau_i(h) = O(h^2)$

neglecting truncation error: $Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1})$

Remark: the deduction is not necessary, ~~the~~.

c) $dy/dx = -\lambda 5\, y^{3.5}$, ~~…~~, ~~…~~ .

Forward order for $\cdot\ y' = f(x,y) = -\lambda y$

$$Y_{i+1} = Y_i - h\lambda Y_i.$$

$$Y_{i+1} = \underbrace{(1 - \lambda h)}_{G} Y_i \quad ; \quad Y_{i+1} = G\, Y_i$$ with amplification factor $\rightarrow G$

The method is stable if $|G| \le 1$, that is, if

$$|1 - \lambda h| \le 1$$

For real λ the stability condition is:

$$-1 \leq 1-\lambda h \leq 1.$$
$$-2 \leq -\lambda h \leq 0$$
$$2 \geq \lambda h \geq 0$$

Thus the stability condition for $h > 0$, $\lambda h \leq 2$

Backward Euler method for $y' = -\lambda y$

$$Y_{i+1} = Y_i - h\lambda \, Y_{i+1}.$$
$$(1 + h\lambda) Y_{i+1} = Y_i$$
$$Y_{i+1} = G Y_i \quad \text{with} \quad G = \frac{1}{1+\lambda h}$$

For $\lambda > 0$ $\quad |G| = \dfrac{1}{1+\lambda h} \quad \forall h > 0 \Rightarrow$ unconditionally stable

d) $f(x,y) = -25y^{3.5}$, $h = 0.1$, $y(0) = 1$. -initial condition

Backward Euler : $Y_{i+1} = Y_i - h \, 25 Y_{i+1}^{3.5}$ $\quad h = 0.1.$

Given Y_i. we have to solve the nonlinear equation : $F(Y_{i+1}) = 0$.

with $F(z) = z - Y_i + 2.5 z^{3.5}.$

$$F'(z) = 1 + 8,75 z^{2.5}$$

Newton method : $\quad z^{k+1} = z^k - \dfrac{F(z^k)}{F'(z^k)}$

For $i = 0$; $Y_0 = 1$

Initial guess is equal to previous step :

$$z^0 = Y_0 = 1$$
$$z^1 = z^0 - \frac{F(z^0)}{F'(z^0)} = 1 - \frac{2,5}{9,75} = 0,74359$$
$$z^2 = z^1 - \frac{F(z^1)}{F'(z^1)} = 0,62179 \rightarrow Y_1 \simeq 0,62179$$

[Remark: with more iterations the backward Euler solution: is $Y_0 = 1$; $Y_1 = 0,594643$, $Y_2 = 0,446242$]

For $i = 1$, $Y_1 = 0,62179$.

$$z^0 = Y_1.$$
$$z^1 = 0,48257$$
$$z^2 = 0,46025 \rightarrow Y_2 \simeq 0,46025$$

e) $Y_0 = 1$;

$$Y_1 = Y_0 + 0,1 \left(-25 Y_0^{3.5}\right) = -1.5$$
$$Y_2 = Y_1 + 0,1 \left(-25 . Y_1^{3.5}\right) = -1.5 + 10,334 i$$

the method is unstable

f) The asimptotic stability analysis can only be done for linear functions, so we will first linearize the function a neighborhood of $y=1$;

Taylor series: $P(y+\Delta y) = F(1) + \Delta y\, F'(1)$

So we can write:

$$f(y) = -25y^{3,5} \approx f(1) + f'(1)(y-1)$$

$$f(y) \approx -25 - 87{,}5(y-1)$$

$$f(y) \approx -62{,}5 - \underbrace{87{,}5}_{\lambda}\, y$$

The Euler method is stable for $0 \leq \lambda h \leq 2$ (for $\lambda > 0$,

That $h \leq \frac{2}{\lambda} \approx 0{,}02286 = h^*$

with numerical experiments

The method is unstable for $h = \frac{1}{10}, \frac{1}{15}, \frac{1}{30} > h^*$

and stable for $h = \frac{1}{45}, \frac{1}{90} < h^*$ confirming the analysis.