NM4PDEs - Exercises ODEs

1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1$ m is the pendulum length and the gravity acceleration is $g = 9.8$ m/s^2.

The position and velocity at time $t = 1$ s are known:

$$\theta(1) = 0.4 \text{ rad} \quad ; \quad \frac{\mathrm{d}\theta}{\mathrm{dt}}(1) = 0 \text{ rad/s}$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0$ s, with 2 and 4 time steps.

b) Using the approximations obtained in *a)*, compute an approximation of the relative error in the solution computed with 2 steps.

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

a) initial position $t = 0$ with Runge-Kutta with 2 and 4 steps

Using Heun's method

$$\boldsymbol{y}^{i+1} = \boldsymbol{y}^{i} + \frac{h}{2}(k_1 + k_2)$$

$$\boldsymbol{k_1} = f(x^i, \boldsymbol{y}^i)$$

$$\boldsymbol{k_2} = f(x^i + h, \boldsymbol{y}^i + k_1 h)$$

Applied to a system of two equations equivalent to the second order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$y_1 = \theta$$

$$y_2 = \frac{d\theta}{dt}$$

This change of variables yields the system

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boldsymbol{f}(x, \boldsymbol{y}) = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \boldsymbol{y}^i$$

Applying Heun's method with two steps $h = -0.5\text{s}$

- Step 1

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \boldsymbol{y}^i$$

$$\boldsymbol{y}^{t=1} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.92 \end{bmatrix}$$

$$\boldsymbol{k_2} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \left(\begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ -3.92 \end{bmatrix} \right) = \begin{bmatrix} 1.96 \\ -3.92 \end{bmatrix}$$

$$\boldsymbol{y}^{t=0.5} \quad = \quad \boldsymbol{y}^{t=1} - \frac{h}{2}(\boldsymbol{k_1} + \boldsymbol{k_2}) \quad = \quad \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

- Step 2

$$\boldsymbol{y}^{t=0.5} = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} = \begin{bmatrix} 1.96 \\ 0.882 \end{bmatrix}$$

$$\boldsymbol{k_2} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \left(\begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ -3.92 \end{bmatrix} \right) = \begin{bmatrix} 1.519 \\ 10.486 \end{bmatrix}$$

$$\boldsymbol{y}^{t=0} \quad = \quad \boldsymbol{y}^{t=0.5} - \frac{h}{2}(\boldsymbol{k_1} + \boldsymbol{k_2}) \quad = \quad \begin{bmatrix} -0.9598 \\ -0.8820 \end{bmatrix}$$

Applying Heun's method with four steps $h = -0.25\text{s}$

Step	$\boldsymbol{y}^i$	$\boldsymbol{k_1}$	$\boldsymbol{k_2}$	$\boldsymbol{y}^{i+1}$
1	$\begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -3.92 \end{bmatrix}$	$\begin{bmatrix} 0.98 \\ -3.92 \end{bmatrix}$	$\begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$
2	$\begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$	$\begin{bmatrix} 0.98 \\ -2.7195 \end{bmatrix}$	$\begin{bmatrix} 1.6599 \\ -0.3185 \end{bmatrix}$	$\begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix}$
3	$\begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix}$	$\begin{bmatrix} 1.3598 \\ 0.5143 \end{bmatrix}$	$\begin{bmatrix} 1.2312 \\ 3.8457 \end{bmatrix}$	$\begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix}$
4	$\begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix}$	$\begin{bmatrix} 0.8147 \\ 3.6882 \end{bmatrix}$	$\begin{bmatrix} -0.1073 \\ 5.6843 \end{bmatrix}$	$\begin{bmatrix} -0.4648 \\ -0.3568 \end{bmatrix}$

b) Relative error

$$y_1^a(t) = A \quad \sin(\omega t + \emptyset)$$

$$d_t y_1^a(t) = y_2^a(t) = A\omega \;\; \cos(\omega t + \emptyset)$$

$$d_{tt} y_1^a(t) = -A\omega^2 \;\; \sin(\omega t + \emptyset)$$

$$\omega^2 = g/L$$

$$y_2^a(t = 1) = A\omega \;\; \cos\left(\sqrt{g/L} + \emptyset\right) = 0$$

$$\emptyset = \left(n + \frac{1}{2}\right)\pi - \sqrt{g/L} \qquad\qquad n = 0,1,2,\ldots$$

$$y_1^a(t = 1) = A \;\; \sin\left(\sqrt{g/L} + \emptyset\right) = 0.4$$

$$A = 0.4$$

$$y_1^a(t = 0) = 0.4 \;\; \sin\left(\left(n + \frac{1}{2}\right)\pi - \sqrt{g/L}\right) = -0.399975$$

$$\epsilon = \left|\frac{y_1 - y_1^a}{y_1^a}\right| = \left|\frac{-0.9598 + 0.399975}{-0.399975}\right| \approx 140\%$$

c) Time step h for error three orders of magnitude smaller

Number of steps	Error ϵ, %
2	140%
4	16%
8	2.2%
10	1.1%