(1) The motion of a non frictional Pendulum is governed by the ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$g = 9.8\ m/s^2$ $\quad L = 1$ $\quad \theta(1) = 0.4$ rad $\quad \frac{d\theta}{dt}(1) = 0$ rad/s

a) Solve the inital boundry value Problem in the interval (0,1) using Second-order Rung kutta method to determine the inital Position at t=0 with 2 and 4 time steps

b) using the approximations obtained in a), compute an approximation of realtive error in the soultion computed with 2 steps.

c) Propose a time step h to obtain an approximation with a reative error three orders of magnitude Smaller.

$$\frac{d\theta_1}{dt} = \theta_2 = \frac{d\theta}{dt} = \theta_1'$$

$$\frac{d\theta_2}{dt} + g\,\theta_1 = 0 \Rightarrow \frac{d\theta_2}{dt} = -g\theta_1$$

$$\frac{d\theta}{dt} = L(t,\theta) = \begin{bmatrix} \theta_2 \\ -g\theta_1 \end{bmatrix}$$

$$\theta_{i+1} = \theta_i + h/2\,[k_1 + k_2]$$

$$k_1 = L(t_i, \theta_i) \qquad k_2 = L(t_i + h, \theta_i + hk_1)$$

$$k_1^{(1)} = \begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}$$

$$\theta^{(0)} = \begin{bmatrix} \theta_1^{(0)} \\ \theta_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - h/2\,[k_1^{(1)} + k_2^{(1)}]$$

$$k_2^{(1)} = L(t_i - h;\ \theta_i - hk_1)$$

$$L = \left(t_i - h,\ \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}\right)$$

$$k_2^{(1)} = \begin{bmatrix} g(0.2) \\ -g(0.4) \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - h/2\,[k_1 + k_2]$$

$$\theta^{(1)} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{0.5}{2} \begin{bmatrix} 1.96 \\ -7.84 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 1.96 \end{bmatrix}$$

$$\theta^{(0)} = \theta^{(1)} - h/2\,[k_1^{(2)} + k_2^{(2)}]$$

$$k_2^{(2)} = L\left(t^{(1)} - h,\ \begin{bmatrix} 0.89 \\ 2.40 \end{bmatrix}\right) = \begin{bmatrix} 2.401 \\ g(0.89) \end{bmatrix}$$

$$\theta^{(2)} = \begin{bmatrix} -1.00025 \\ 0 \end{bmatrix}$$

4 step $\quad h = 0.25$

$$\frac{d\theta}{dt} = L(t, \theta) = \begin{bmatrix} \theta_2 \\ -g\theta_1 \end{bmatrix}$$

$$\theta^{(0)} = \begin{bmatrix} \theta_1(1) \\ \frac{dg(1)}{dt} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

Second order rung kuttah

$$k_2^{(1)} = L\left(t^{(0)} - h,\ \theta^{(0)} - hk_1^{(1)}\right)$$

$$k_2^{(1)} = \begin{bmatrix} g(0.1) \\ -g(0.4) \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - h/2\,[k_1^{(1)} + k_2^{(1)}] = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$$

$$\theta^{(2)} = \theta^{(1)} - h/2\,[k_1^{(2)} + k_2^{(2)}]$$

$$k_2^{(2)} = L\left(t^{(1)} - h,\ \begin{bmatrix} 0.0325 \\ 1.6599 \end{bmatrix}\right) = \begin{bmatrix} 1.6599 \\ -g(0.0325) \end{bmatrix}$$

$$\Theta^{(2)} = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - \begin{bmatrix} 0.3299 \\ -0.3001 \end{bmatrix} = \begin{bmatrix} -0.0524 \\ 1.2801 \end{bmatrix}$$

$$k_2^{(3)} = L\left(t^{(2)} - h, \Theta^{(2)} - h k_1^{(3)}\right)$$

$$\Rightarrow k_2^{(3)} = \begin{bmatrix} 1.1515 \\ 3.6505 \end{bmatrix}$$

$$\Theta^{(3)} = \Theta^{(2)} - \frac{h}{2}\left[k_1^{(3)} + k^{(3)}\right]$$

$$\Theta^{(3)} = \begin{bmatrix} -0.3564 \\ 0.7595 \end{bmatrix}$$

$$\Theta^{(4)} = \Theta^{(3)} - \frac{h}{2}\left[k_1^{(4)} + k_2^{(4)}\right]$$

$$k_1^{(4)} = \begin{bmatrix} 0.7595 \\ 3.4932 \end{bmatrix}$$

$$k_2^{(4)} = L\left(t^{(4)} - h, \Theta^{(4)} - h k_1^{(4)}\right)$$

$$\Rightarrow L\left[t^{(4)} - h, \begin{bmatrix} -0.5163 \\ -0.1138 \end{bmatrix}\right]$$

$$k_2^{(4)} = \begin{bmatrix} -0.11380 \\ 5.35398 \end{bmatrix}$$

$$\Theta^{(4)} = \begin{bmatrix} -0.4372 \\ -0.3464 \end{bmatrix}$$

② Consider the inital value Problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0, 1) \qquad y(0) = 1$$

a) Solve the inital value Problem using the Euler method with step $h = 0.25$

b) Compute the soultion using Heun method with step h such that the computational cost is equivacent to computational cost in a

$$\frac{dy}{dx} = y - x^2 + 1 \Rightarrow f(x) = y - x^2 + 1 \qquad h = 0.25$$

a)

$i=0 \Rightarrow y_1 = y_0 + f(x_0, y_0)h \Rightarrow y_1 = 1 + (1 - (0)^2 + 1) \times 0.25 = 1.5$

$i=1 \Rightarrow y_2 = y_1 + f(x_1, y_1)h \Rightarrow y_2 = 1.5 + (1.5 - (0.25)^2 + 1) \times 0.25 = 2.10$

$i=2 \Rightarrow y_3 = y_2 + f(x_2, y_2)h \Rightarrow y_3 = 2.10 + (2.10 - (0.5)^2 + 1) \times 0.25 = 2.81$

$i=3 \Rightarrow y_4 = y_3 + f(x_3, y_3)h \Rightarrow y_4 = 2.81 + (2.81 - (0.75)^2 + 1) \times 0.25 = 3.62$

b) $$y_{i+1} = y_i + \frac{f(x_i) + f(x_{i+1})h}{2}$$

$$y_1^{(1)} = 1 + \frac{\left[(1 - 0^2 + 1) + (1.5) - (0.5)^2 + 1)\right]}{2} \times 0.5 = 1.87$$

$$y_2^{(2)} = 1 + \frac{\left[(1 - 0^2 + 1) + (1.87) - (1)^2 + 1)\right]}{2} \times 0.5 = 2.67$$

③ The ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

$$Y_{i+1} = Y_i + h f(x_i, Y_i)$$

$$Y_{i+1} = Y_i + h\lambda Y_i$$

$$Y_{i+1} = G Y_i$$

$$Y_{i+1} = Y_i + h\lambda Y_{i+1}$$

$$Y_{i+1} = Y_i + h \frac{dy}{dx}(x_i) + \frac{h^2}{2} \frac{d^2y}{dx^2}(x_i) + O(h^3)$$

$$Y_{i-1} = Y_i - h \frac{dy}{dx}(x_i) + \frac{h^2}{2} \frac{d^2y}{dx^2}(x_i) + O(h^3)$$

$$\frac{dy}{dx}(x_i) = \frac{Y_{i+1} - Y_{i-1}}{2h} - \tau_i h$$

$$Y_{i+1} - Y_i = \int_{x_i}^{x_{i+1}} f(x, y(x))\,dx$$

$$Y_{i+1} = Y_i + \frac{h}{2}\left[f(x_i, Y_i) + f(x_{i+1}, Y_{i+1}) \right.$$

$$Y_{i+1} = Y_i + h f(x_i, Y_i)$$

$$Y_{i+1} = Y_i + h/2 \left[f(x_i, Y_i) + f(x_{i+1}, Y^*_{i+1}) \right.$$