Numerical Methods for PDE ARJUN AJAY

Exercise 3

2. $u_t + a u_x = 0 \qquad x \in (0,1),\ t \geq 0,\ a > 0$

initial condition: $u(x,0) = \sin(2\pi x)$

Periodic boundary condition: $u(0,t) = u(1,t)$

a) Implicit scheme 1st order in time and space.

Using 1st order finite difference approximation for time and space

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \cdot \left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} \right) = 0$$

$$\Rightarrow \quad u_i^{n+1} - u_i^n = \frac{a \Delta t}{\Delta x} \left(u_i^{n+1} - u_{i+1}^{n+1} \right)$$

$$r = \frac{a \Delta t}{\Delta x}$$

$$\Rightarrow \quad (1-r) u_i^{n+1} + r u_{i+1}^{n+1} = u_i^n$$

b) In each time step, assuming there are $m+1$ nodes from 0 to m.

$$m \text{ equations} \left\{ \begin{array}{l} (1-r) u_0^{n+1} + r u_1^{n+1} = u_0^n \\ (1-r) u_1^{n+1} + r u_2^{n+1} = u_1^n \\ \vdots \\ (1-r) u_{m-1}^{n+1} + r u_m^{n+1} = u_{m-1}^n \end{array} \right.$$

Boundary condition.

$$u_0^{n+1} = u_m^{n+1}$$

Therefore we have $m+1$ equations and $m+1$ unknowns at each time step which can be solved. Thus the periodic boundary condition is treated with the other m equations as a system of $(m+1)$ linear equations.

c) Direct Method - Gauss Elimination

Indirect Method - Gauss Seidel method

d) Fill in Matrix

$$
\begin{bmatrix}
1-\gamma & \gamma & 0 & \cdots & & & \\
0 & 1-\gamma & \gamma & 0 & \cdots & & \\
\vdots & 0 & 1-\gamma & \gamma & 0 & \cdots & \\
\vdots & & & \ddots & \ddots & \ddots & \\
\vdots & & & & & 1-\gamma & \gamma \\
\gamma & 0 & \cdots & & & \cdots & 1-\gamma
\end{bmatrix}
$$

4. $u_t = \nu u_{xx} + \sigma u \quad \text{in } x \in (0,1), \ t > 0$

a) Explicit finite difference scheme

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

The FDE

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} \right) \cancel{\Delta t} + \sigma u_i^n$$

$$\Rightarrow \quad u_i^{n+1} = r\left(u_{i-1} - 2u_i^n + u_{i+1}^n \right) + \sigma \Delta t u_i^n + u_i^n$$

$$u_i^{n+1} = r u_{i-1}^n + (1 + \sigma \Delta t - 2r) u_i^n + r u_{i+1}^n$$

Initial conditions

$$u(x,0) = \begin{cases} 0, & x < 1/4 \\ 4x-1, & 1/4 \leq x < 1/2 \\ -4x+3, & 1/2 \leq x < 3/4 \\ 0, & 3/4 \leq x \end{cases}$$

Boundary conditions

Dirichlet b.c at $x=0$

$$u(0,t) = 0. \quad \Rightarrow \quad u_0^n = 0 \quad \forall n.$$

Neumann boundary condition at $x=1$

$$u_x(1,t) = 0. \longrightarrow ①$$

At each time step, we have $m+1$ nodes (0 to m).

Take a fictitious node $m+1$.

Applying central difference for the derivative at m.

$$\left.\frac{\partial u}{\partial x}\right|_m^n = \frac{u_{m+1} - u_{m-1}}{2\Delta x} = 0 \quad (\text{from } ①)$$

$\Rightarrow \quad u_{m+1} = u_{m-1}$ from the neumann b.c.

Substituting in the FTCS scheme at the node m

$$u_m^{n+1} = r u_{m-1}^n + (1 + \sigma\Delta t - 2r) u_m^n + r u_{m-1}^n$$

$$= 2r u_{m-1}^n + (1 + \sigma\Delta t - 2r) u_m^n$$

Therefore, we can find the u values at the boundary $x=1$.

b) For $\sigma = 0$,

$$u_t = \nu u_{xx}.$$

$\therefore$ the FTCS scheme is :

$$u_i^{n+1} = r u_{i-1} + (1-2r) u_i^n + r u_{i+1}^n$$

For $\nu = 0$, $\Rightarrow r = 0$.

the FTCS scheme is

$$u_i^{n+1} = (1+\sigma\Delta t) u_i^n$$

c) $\Delta x = 0.25$.

Therefore at each timestep, we have 5 nodes in the x-axis.

As the initial condition is given, we know the initial u values at $t=0$.

$$u_0^0 = 0$$

$$u_1^0 = 0$$

$$u_2^0 = 1$$

$$u_3^0 = 0$$

$$u_4^0 = 0$$

$$r = \frac{\nu\Delta t}{\Delta x^2} = 0.16 \qquad \sigma\Delta t = -0.01$$

At the 1st time step.

$$u_0^1 = 0 \text{ (b.c)}$$

$$u_1^1 = r u_0^0 + (1+\sigma\Delta t - 2r) u_1^0 + r u_2^0$$

$$= 0.16.$$

Similarly we have,

$u_2^1 = 0.67$

$u_3^1 = 0.16$

$u_4^1 = 2ru_3^0 + (1+\sigma\Delta t - 2r)u_4^0 = 0 \quad (B.C)$

For the 2nd time step.

$u_0^2 = 0 \quad (b.c)$

$u_1^2 = 0.2144$

$u_2^2 = 0.5001$

$u_3^2 = 0.2144$

$u_4^2 = 0.0512$

The results obtained seem reasonable. We get a symmetric profile for the u.

d) Implicit F.D scheme.

Let us choose the Backward in time, Central in Space scheme (BTCS)

$$\therefore \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu\left(\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}\right) + \sigma u_i^{n+1}$$

$$\Rightarrow \quad u_i^{n+1} - u_i^n = r\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right) + \sigma\Delta t\, u_i^{n+1}$$

where $r = \dfrac{\nu\Delta t}{\Delta x^2}$.

$$\Rightarrow \quad -r u_{i-1}^{n+1} + (1+2r-\sigma\Delta t) u_i^{n+1} - r u_{i+1}^{n+1} = u_i^n$$

with same initial conditions.

and b.c. $u_0^{n+1} = 0$. and $u_{m+1}^{n+1} = u_{m-1}^{n+1}$ (Using a fictitious node for the neumann b.c at m).

For $i=1$.

m equations
$$\begin{cases} -r u_0^{n+1} + (1+2r-\sigma\Delta t) u_1^{n+1} - r u_2^{n+1} = u_1^n \to (1) & (u_0^{n+1}=0, \text{ b.c}) \\ -r u_1^{n+1} + (1+2r-\sigma\Delta t) u_2^{n+1} - r u_3^{n+1} = u_2^n \to (2) & \\ \vdots & \\ -r u_{m-2}^{n+1} + (1+2r-\sigma\Delta t) u_{m-1}^{n+1} - r u_m^{n+1} = u_{m-1}^n & \\ -2r u_{m-1}^{n+1} + (1+2r-\sigma\Delta t) u_m^{n+1} = u_m^n & (\text{From neumann b.c at node } m) \end{cases}$$

∴ the matrix representing this linear system of equations are:

$$\begin{bmatrix} 1+2r-\sigma\Delta t & -r & 0 & \cdots & \cdots & 0 \\ -r & 1+2r-\sigma\Delta t & -r & & & \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & 0 & -r & 1+2r-\sigma\Delta t & -r \\ 0 & \cdots & \cdots & 0 & -2r & 1+2r-\sigma\Delta t \end{bmatrix}$$

Since the matrix is tridiagonal we can do LU decomposition using for example doolittle algorithm.
We can also use the thomas algorithm to solve the linear system of equations.