1. $f(x) = x^3 + 2x^2 + 10x - 12 = 0$

Initial approximation $x_0 = \sqrt[3]{20} = 2.7144$

$f'(x) = 3x^2 + 4x + 10$

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$$

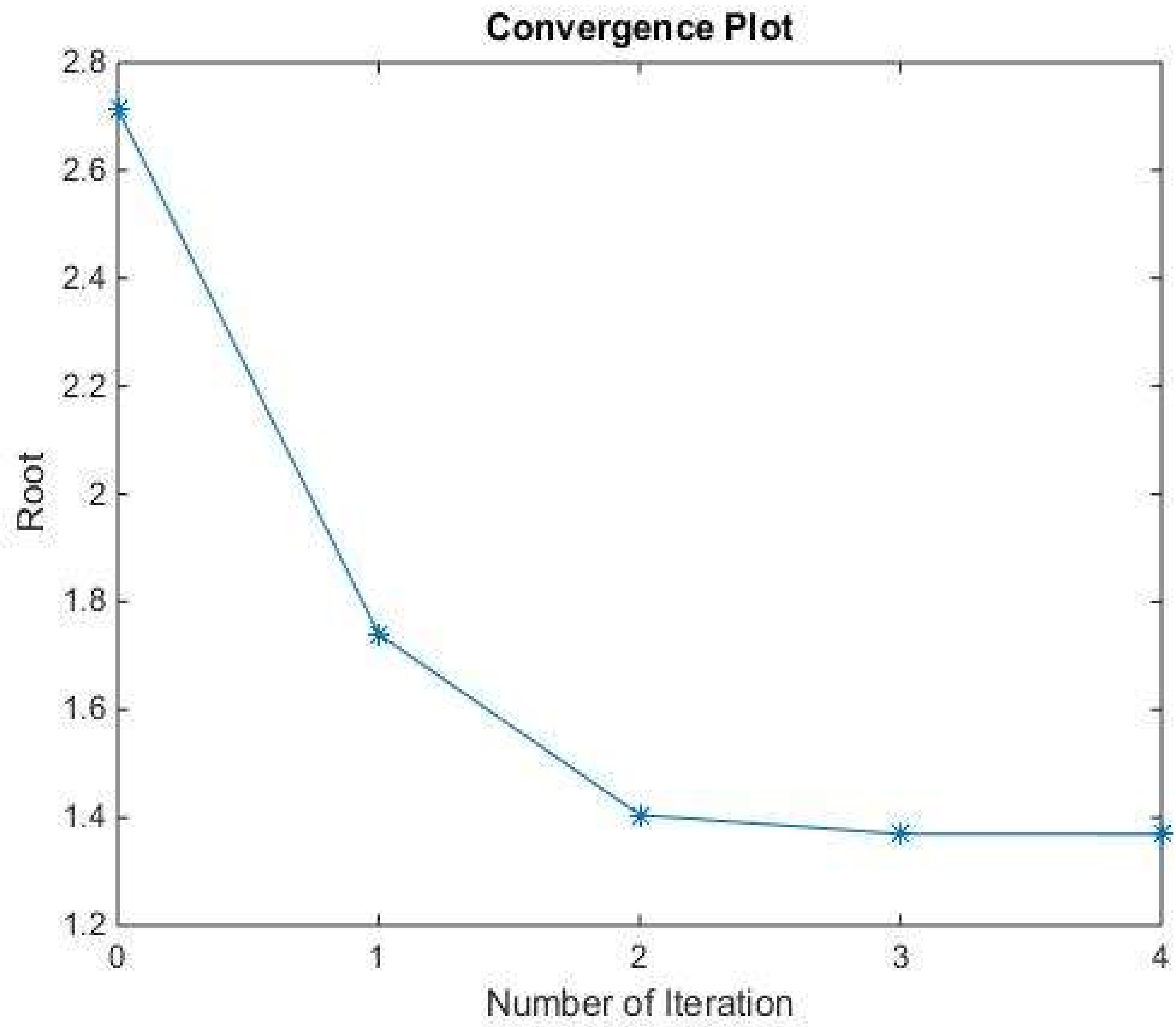
$\therefore$ $x_1 = 1.7396$

$x_2 = 1.4050$

$x_3 = 1.3692$

$x_4 = 1.3688$

Therefore, after 4 iterations, the root of $f(x)$ using newton method is 1.3688


Convergence Plot
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
Root
0
1
2
3
4
Number of Iteration

5. 3^{rd} order numerical quadrature in (0,1)

a) For a third-order quadrature, $n=1$.

$\therefore$ Minimum number of integration points $= n+1 = 2$

For a gauss quadrature in interval $(-1,1)$, the quadrature points are -0.5773 with weight 1 and 0.5773 with weight 1.

Therefore for the interval $(0,1)$.

$$x_i = \left(\frac{b-a}{2}\right) z_i + \left(\frac{a+b}{2}\right) \quad \text{where} \quad a=0 \text{ and } b=1.$$

$$\therefore x_1 = \frac{1}{2} \times -0.5773 + \frac{1}{2} = 0.21135$$

$$x_2 = \frac{1}{2} \times 0.5773 + \frac{1}{2} = 0.78865$$

The new weights are $w_1 = \frac{(b-a)}{2} w = \frac{1}{2}$

$$w_2 = \frac{1}{2}.$$

b) Third order quadrature must be able to exactly integrate the cubic polynomial : $Ax^3 + Bx^2 + cx + D = f(x)$

$$\therefore \int_0^1 Ax^3 + Bx^2 + cx + D = w_0 f(1/4) + w_1 f(1/2) + w_2 f(3/4) + w_3 f(1)$$

$$\Rightarrow \left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{cx^2}{2} + Dx\right]_0^1 = w_0\left(\frac{A}{64} + \frac{B}{16} + \frac{C}{4} + D\right) + w_1\left(\frac{A}{8} + \frac{B}{4} + \frac{C}{2} + D\right)$$
$$+ w_2\left(\frac{27A}{64} + \frac{9B}{16} + \frac{3C}{4} + D\right) + w_3(A + B + C + D)$$

$$\Rightarrow A\left(\frac{w_0}{64}+\frac{w_1}{8}+\frac{27}{64}w_2+w_3-\frac{1}{4}\right)+B\left(\frac{w_0}{16}+\frac{w_1}{4}+\frac{9w_2}{16}+w_3-\frac{1}{3}\right)$$

$$+C\left(\frac{w_0}{4}+\frac{w_1}{2}+\frac{3w_2}{4}+w_3-\frac{1}{2}\right)+D\left(w_0+w_1+w_2+w_3-1\right)=0$$

Solving the 4 equations we get :

$w_0 = 0.667$, $w_1 = -0.333$ $w_2 = 0.667$

$w_3 = 0$.

6. a) The order of the polynomial that is exactly integrated is $2n+1$

b) $n=2 \Rightarrow$ the order of the polynomial that can be exactly integrated is $2n+1 = 5$.

(i) $\sin x$ is not a polynomial, so cannot be integrated exactly

(ii) x^3 can be integrated exactly

(iii) x^4, can be integrated exactly

(iv) $x^{5.5}$ is not a polynomial and so cannot be exactly integrated.

7. $\int_0^1 12x\,dx$.

(i) Trapezoidal rule over 2 uniform intervals.

therefore from 0 to 0.5 and then from 0.5 to 1.

$h = 0.5$

$$\therefore \int_0^1 12x\,dx = \frac{h}{2}\left(f(0) + f(0.5)\right) + \frac{h}{2}\left(f(0.5) + f(1)\right)$$

$$= \frac{0.5}{2}\left(0 + 6\right) + \frac{0.5}{2}\left(6 + 12\right)$$

$$= \frac{1}{4} \times 24 = \underline{\underline{6}}$$

Error $(E) = 0$.

(ii) Simpsons rule over 2 uniform interval.

For the first interval between 0 to 0.5.

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$. and $h = 0.25$.

$$\therefore I_1 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{1}{12}\left[0 + 12 + 6\right] = \frac{3}{2}$$

For the second interval between 0.5 and 1

$x_0 = 0.5$, $x_1 = 0.75$, $x_2 = 1$ and $h = 0.25$

$$\therefore I_2 = \frac{1}{12}\left[6 + 36 + 12\right] = \frac{9}{2}$$

$\therefore$ $I = I_1 + I_2 = \underline{\underline{6}}$ and $E = 0$.

$\int_0^1 (5x^3 + 2x)\, dx$ $\qquad f(x) = 5x^3 + 2x$

(i) Trapezoidal rule over 2 uniform intervals.

$\therefore \quad I = \frac{0.5}{2}(0 + 1.625) + \frac{0.5}{2}(1.625 + 7)$

$= 2.5625$

$E = 2\times -\frac{h^3}{12} f''(\mu) = 2\times -\frac{(0.5)^3}{12} \times 30\mu = -0.3125\,\mu \times 2 = -0.6250\mu$

where $\mu \in (0,1)$

(ii) Simpsons rule over 2 uniform interval.

$I_1 = \frac{0.25}{3}\left(f(0) + 4 f(0.25) + f(0.5)\right)$

$= \frac{1}{12}(0 + 2.3124 + 1.625)$

$= 0.3281$

$I_2 = \frac{0.25}{3}\left(f(0.5) + 4 f(0.75) + f(1)\right)$

$= \frac{1}{12}(1.625 + 14.4376 + 7)$

$= 1.9219$

$\therefore \quad I = I_1 + I_2 = \underline{\underline{2.25}}$ $\qquad E = -2\times \frac{h^5}{90} f^4(\mu) = 0$

The trapezoidal rule can exactly integrate a linear function and therefore gives an error in the case of cubic function.
The simpsons rule can exactly integrate a cubic functions and therefore gives no error.

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10. $\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$

Applying simpsons rule in the x direction, $f(x) = 9x^3 + 8x^2$

$$I_1 = \frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right] \quad \text{where } h = 0.5.$$

$$= \frac{1}{6}\left[0 + 12.5 + 17\right] = 4.9167$$

Applying simpsons rule in the y direction, $f(y) = y^3 + y$

$$I_2 = \frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right]$$

$$= \frac{1}{6}\left[0 + 2.5 + 2\right] = 0.75.$$

$$\therefore \quad I = I_1 \times I_2 = 3.6875.$$

The approximation is behaving as expected with no error.