10/11/15

HOME WORK - ①

Anil Bettadahalli Channakeshava

$$-u'' = f \quad \text{in }]0,1[$$

or

$$u'' + f(x) = 0 \;,\;]0,1[$$

With B.C $u(0) = 0$, $u(1) = \alpha$

* Strong form to weak form : using (WRM)

$$\int_0^1 w_i \left[u'' + f(x)\right] dx = 0$$

$$\underbrace{\int_0^1 w_i\, u''\, dx}_{} + \int_0^1 w_i f(x)\, dx = 0$$

Integrating by parts

$$\int_0^1 w_i\, u''\, dx = w_i\, u'\Big|_0^1 - \int_0^1 \frac{dw_i}{dx}\left(\frac{du}{dx}\right) dx$$

or

$$\int_0^1 \frac{dw_i}{dx}\frac{du}{dx}\, dx = \int_0^1 w_i f(x)\, dx + w_i \left(\frac{du}{dx}\right)\Bigg|_0^1$$

Let $R = -\frac{du}{dx}$ (Reaction force on Direchlit Boundary at $(x=0)$) (unknown)

R & $\bar{R}$ is unknown

& $\cdot\; \bar{R} \rightarrow$ Reaction force at Neumann Boundary = unknown at $(x=l)$

$$\left.\begin{aligned}\int_0^1 \frac{dw_i}{dx}\frac{du}{dx}\, dx = \int_0^1 w_i f(x)\, dx + \left[w_i R\right]_0 - \left[w_i \bar{R}\right]_1\end{aligned}\right\} \text{weak form}$$

Approximation of unknown, we have

$$u \simeq u^h = \sum_{j=1}^{n} N_j(x)\, u_j$$

$$\int_0^1 \frac{dw_i}{dx} \frac{d}{dx}\left[\sum_{j=1}^{n} N_j(x)\, u_j\right] dx = \int_0^1 w_i f(x)\, dx + [w_i R]_0 - (w_i \bar{R}]_1$$

Using Galerkin Method

$$w_i = N_i$$

$$\underbrace{\int_0^1 \frac{dw_i}{dx} \sum_{j=1}^{n} \frac{d}{dx}[N_j(x)]}_{K_{ij}} \underbrace{u_j}_{a} \, dx = \underbrace{\int_0^1 N_i f(x)\, dx + [N_i R]_0 - (N_i \bar{R}]_1}_{f_i}$$

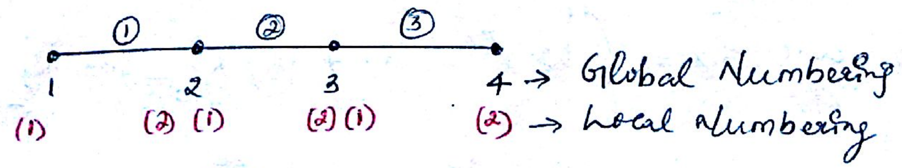
$$\underline{\underline{K}}\, \underline{a} = \underline{f}$$

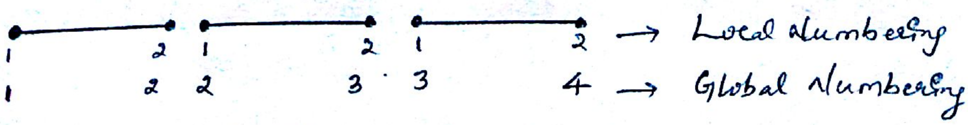
$$K_{ij} = \frac{dN_i}{dx} \frac{d}{dx}[N_j(x)]$$

$$a = u_j$$

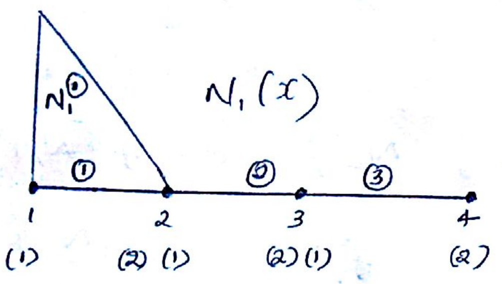
$$f_i = N_i f(x) + [N_i R]_0 - (N_i \bar{R}]_1$$

✱ Linear System of Equations to be solved

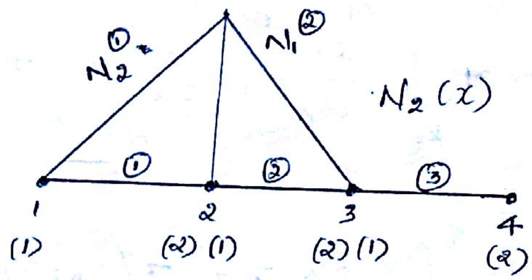




Shape functions

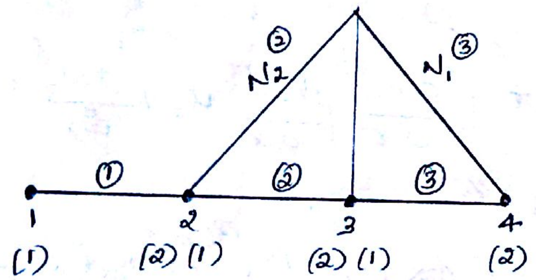


Global	Local	
$N_1 =$	$N_1^{(1)}$	$0 \le x \le 1/3$
$N_1 =$	0	$1/3 \le x \le 2/3$
$N_1 =$	0	$2/3 \le x \le 1$

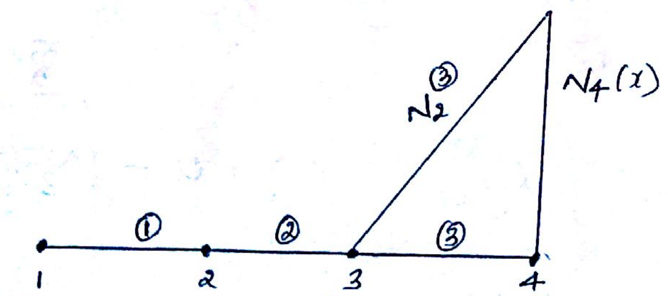


$\cancel{N_2 = 0}$
$\cancel{N_2 = N_2^{(1)}}$

$$
\left.\begin{aligned}
N_2 &= N_2^{(1)} \\
N_2 &= N_1^{(2)} \\
N_2 &= 0
\end{aligned}\right\}
\begin{aligned}
&0 \le x \le 1/3 \\
&1/3 \le x \le 2/3 \\
&2/3 \le x \le 1
\end{aligned}
$$



$$
\left.\begin{aligned}
N_3 &= 0 \\
N_3 &= N_2^{(2)} \\
N_3 &= N_1^{(3)}
\end{aligned}\right\}
\begin{aligned}
&0 \le x \le 1/3 \\
&1/3 \le x \le 2/3 \\
&2/3 \le x \le 1
\end{aligned}
$$



$$
\left.\begin{aligned}
N_4 &= 0 \\
N_4 &= 0 \\
N_4 &= N_2^{(3)}
\end{aligned}\right\}
\begin{aligned}
&0 \le x \le 1/3 \\
&1/3 \le x \le 2/3 \\
&2/3 \le x \le 1
\end{aligned}
$$

From weak form, we have

$$
\underbrace{\int_0^1 \frac{dN_i}{dx}\left[\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 + \frac{dN_3}{dx}u_3 + \frac{dN_4}{dx}u_4\right]dx}_{\text{Global Solution System}} = \int_0^1 N_i f(x)\,dx + \left[N_i R\right]_0 - \left[N_i \bar{R}\right]_1
$$

$i = 1, 2, 3, \ldots$

Local System: (Global Solution written in Local shape function)

for $i = 1$

$$\int_0^{1/3} \frac{dN_1^{(1)}}{dx}\left[\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right]dx = \underbrace{\int_0^{1/3} N_1^{(1)} f(x)\,dx}_{f_1^{(1)}} + \overbrace{[N_1 R]_0}^{R} - \cancel{[N_1 \bar{R}]_1}^{\to 0}$$

for $i = 2$

$$\int_0^{1/3} \frac{dN_2^{(1)}}{dx}\left[\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right]dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx}\left[\frac{dN_1^{(2)}}{dx}u_2 + \frac{dN_2^{(2)}}{dx}u_3\right]dx$$

$$= \underbrace{\int_0^{1/3} N_2^{(1)} f(x)\,dx}_{f_2^{(1)}} + 0 + \overbrace{\int_{1/3}^{2/3} N_1^{(2)} f(x)\,dx}^{f_1^{(2)}} + 0$$

for $i = 3$

$$\int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx}\left[\frac{dN_1^{(2)}}{dx}u_2 + \frac{dN_2^{(2)}}{dx}u_3\right]dx + \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx}\left[\frac{dN_1^{(3)}}{dx}u_3 + \frac{dN_2^{(3)}}{dx}u_4\right]dx$$

$$= \underbrace{\int_{1/3}^{2/3} N_2^{(2)} f(x)\,dx}_{f_2^{(2)}} + 0 + \underbrace{\int_{2/3}^{1} N_1^{(3)} f(x)\,dx}_{f_1^{(3)}} + 0$$

for $i = 4$

$$\int_{2/3}^{1} \frac{dN_2^{(3)}}{dx}\left[\frac{dN_1^{(3)}}{dx}u_3 + \frac{dN_2^{(3)}}{dx}u_4\right]dx = \underbrace{\int_{2/3}^{1} N_2^{(3)} f(x)\,dx}_{f_2^{(3)}} + 0 - \overbrace{[N_3 \bar{R}]_1}^{\bar{R}}$$

The above expression can be written in matrix form as,

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = \alpha \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + R \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} - \bar{R} \end{bmatrix}$$

$\longrightarrow$ (i)

∴ Stiffness matrix for 'n' 2 noded element

$$K = \begin{bmatrix} K_{11}^{①} & K_{12}^{①} & & & & 0 \\ K_{21}^{①} & K_{22}^{①} + K_{11}^{②} & K_{12}^{②} & & & \\ & K_{21}^{②} & K_{22}^{②} + K_{11}^{③} & K_{12}^{③} & & \\ & & K_{21}^{③} & K_{22}^{③} + K_{11}^{④} & K_{12}^{④} & \\ & & & \vdots & & \\ & 0 & & K_{22}^{n-1} + K_{11}^{n} & K_{12}^{n} & \\ & & & K_{21}^{n} & K_{22}^{n-1} & \end{bmatrix}_{n \times n}$$

Global External force Vector

$$\underline{f} = \begin{bmatrix} f_1^{①} + R \\ f_2^{①} + f_1^{②} \\ f_2^{②} + f_1^{③} \\ \vdots \\ f_2^{(n-2)} + f_1^{(n-1)} \\ f_2^{n-1} - \bar{R} \end{bmatrix}_{n \times 1}, \quad a = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}_{n \times 1}$$

$$u = u^h = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

Now, we have

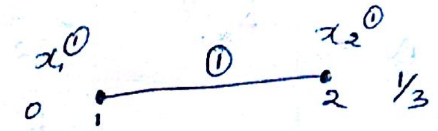
$$K_{ij} = \int_{l^{e}} \frac{dN_i^{e}}{dx} \frac{dN_j^{e}}{dx} \, dx, \quad f_i = \int_{l^{e}} N_i^{e} f(x) \, dx$$

We know that :

$$u^h(x) = a_0 + a_1(x) \quad \text{[polynomial approximation]}$$

③ FE Approximation u^h for $n=3$, $f(x)=\sin x$, $\alpha=3$

for $e=1$, element ①



$$K_{11}^{(e)} = \int_{l^{(e)}} \left(\frac{-1}{l^e}\right)\cdot\left(\frac{-1}{l^e}\right) dx$$

$$K_{11}^{(e)} = \frac{1}{l^e} = K_{22}^{(e)}$$

$$K_{12}^{(e)} = \int_{l^{(e)}} \left(\frac{-1}{l^{(e)}}\right)\left(\frac{1}{l^{(e)}}\right) dx = \frac{-1}{l^{(e)}} = K_{21}^{(e)}$$

$$K_{11}^{(1)} = K_{22}^{(1)} = \int_0^{1/3} \left(\frac{-1}{1/3}\right)\left(\frac{-1}{1/3}\right) dx = 3/1$$

$$K_{21}^{(1)} = K_{12}^{(1)} = \int_0^{1/3} \left(\frac{-1}{1/3}\right)\left(\frac{1}{1/3}\right) dx = -3/1$$

$$f_1^{(1)} = \int_0^{1/3} \left(\frac{x_2^{(1)} - x}{1/3}\right) \sin x \, dx = 3\int_0^{1/3} \left(x_2^{(1)} \sin x - x \sin x\right) dx$$

$$= 3\int_0^{1/3} \left[\tfrac{1}{3}\sin x - x \sin x\right] dx$$

$$= \left[-\cos x \Big|_0^{1/3}\right] - 3\left(-x\cos x + \sin x\right)\Big|_0^{1/3}$$

$$= 0.055043 - 3 \times 0.012209$$

$$\boxed{f_1^{(1)} = 0.0184160}$$

$$f_2^{(1)} = \int_0^{1/3} \left(\frac{x - x_1^{(1)}}{1/3}\right) \sin x \, dx$$

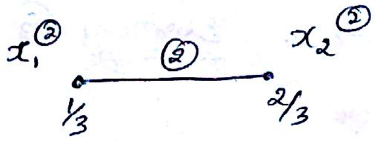
$$= 3 \int_0^{1/3} \left[x \sin x - \cancelto{0}{x_1^{(1)} \sin x}\right] dx \qquad (x_1^{(1)} = 0)$$

$$= 3 \int_0^{1/3} (x \sin x - 0)\, dx$$

$$= 3 \times 0.012209$$

$$\boxed{f_2^{(1)} = 0.036627}$$

* For $e = 2$, Element (2)



$$K_{11}^{(2)} = K_{22}^{(2)} = 3$$

$$K_{12}^{(2)} = K_{21}^{(2)} = -3$$

$$f_1^{(2)} = \int_{1/3}^{2/3} \left(\frac{x_2^{(2)} - x}{1/3}\right) \sin x \, dx$$

$$= 3 \int_{1/3}^{2/3} \left(x_2^{(2)} \sin x - x \sin x\right) dx$$

$$= 3 \int_{1/3}^{2/3} \left[\tfrac{2}{3} \sin x - x \sin x\right] dx$$

$$= -\cos x \Big|_{1/3}^{2/3} - 3\left[-x \cos x + \sin x\right]\Big|_{1/3}^{2/3}$$

$$= 2\,[0.15907] - 3\,[0.082236]$$

$$\boxed{f_1^{(2)} = 0.071432}$$

$$f_2^{②} = \int_{1/3}^{2/3} \left[\frac{x - x_1^{②}}{1/3}\right] \sin x \, dx$$

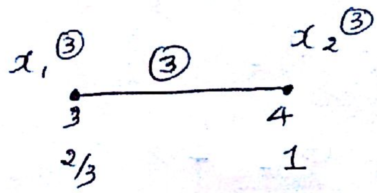
$$= 3 \int_{1/3}^{2/3} \left[x \sin x - \tfrac{1}{3} \sin x\right] dx$$

$$= 3 \left[\left[-x \cos x + \sin x \right]\Big|_{1/3}^{2/3} + \tfrac{1}{3} \cos x \Big|_{1/3}^{2/3} \right]$$

$$= 3(0.082236) - 0.15907$$

$$\boxed{f_2^{②} = 0.087638}$$

* <u>For e = 3</u> , <u>Element ③.</u>



$$f_1^{③} = \int_{2/3}^{1} \left(\frac{x_2^{③} - x}{1/3}\right) \sin x \, dx$$

$$= 3 \int_{2/3}^{1} \left(x_2^{③} \sin x - x \sin x\right) dx$$

$$= 3 \int_{2/3}^{1} (\sin x - x \sin x) \, dx$$

$$= 3 \int_{2/3}^{1} \sin x \, dx - 3 \int_{2/3}^{1} x \sin x \, dx$$

$$= 3 (-\cos x)\Big|_{2/3}^{1} - 3 (-x \cos x + \sin x)\Big|_{2/3}^{1}$$

$$= (3 \times 0.24558) - 3(0.20672)$$

$$\boxed{f_1^{③} = 0.116594}$$

$$f_2^{(3)} = \int_{2/3}^{1} \left(\frac{x - x_1^{(3)}}{1/3} \right) \sin x \, dx$$

$$= 3 \int_{2/3}^{1} x \sin x - \cancel{3} \int_{2/3}^{1} \frac{2}{\cancel{3}} \sin x \, dx$$

$$= 3 \times \left[-x \cos x + \sin x \right]\Big|_{2/3}^{1} - 2 \left[-\cos x \right]_{2/3}^{1}$$

$$= 3 \times 0.20672 - 2(0.2455)$$

$$\boxed{f_2^{(3)} = 0.12899}$$

Substituting all these values in Eqn (i)

$$3 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0184160 + R \\ 0.036627 + 0.071432 \\ 0.087638 + 0.116594 \\ 0.12899 - \bar{R} \end{bmatrix}$$

From data, $u(0) = u_1 = 0$

$u(1) = u_4 = \alpha = 3$

Now, we have

$$3 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184160 + R \\ 0.036627 + 0.071432 \\ (0.108059) \\ 0.087638 + 0.116594 \\ (0.204232) \\ 0.12899 - \bar{R} \end{bmatrix}$$

To solve for u_2 & u_3

$$0 + 6u_2 - 3u_3 = 0.108059$$

$$(0 - 3u_2 + 6u_3 - 9 = 0.204232) \times 2$$

$$\cancel{6u_2} - 3u_3 = 0.108059$$

$(+)$

$$-\cancel{6u_2} + 12u_3 = 18.40846$$

$$9u_3 = 18.5165246$$

$$\Rightarrow \boxed{u_3 = 2.0573}$$

$$6u_2 - 3(2.0573) = 0.108059$$

$$\boxed{u_2 = 1.0467}$$

$$\therefore \boxed{\begin{aligned} u_1 &= 0 \\ u_2 &= 1.0467 \\ u_3 &= 2.0573 \\ u_4 &= 3 \end{aligned}} \rightarrow \text{FEM Solution}$$

Find unknowns : R & $\bar{R}$ [Reaction forces]

$$3(u_1 - u_2) = 0.0184160 + R$$

$$-3u_2 = 0.0184160 + R \qquad u_1 = 0$$

$$-3(1.0467) = 0.0184160 + R$$

$$\Rightarrow \boxed{R = -3.1585}$$

$3(-u_3 + u_4) = 0.12899 - \bar{R}$

$3(-2.0573 + 3) = 0.12899 - \bar{R}$

$\Rightarrow \boxed{\bar{R} = -2.6991}$

* From Data, Given Exact solution is :-

$u(x) = \sin x + (3 - \sin 1)x$

→ put $x = 0$

$u(0) = \sin 0 + (3 - \sin 1) 0$

$\boxed{u(0) = 0}$

→ put $x = 1/3$

$u(1/3) = \sin(1/3) + (3 - \sin 1) 1/3$

$= 0.327194 + (3 - 0.8414) 1/3$

$\boxed{u(1/3) = 1.04670}$

→ put $x = 2/3$

$u(2/3) = \sin(2/3) + (3 - \sin 1) 2/3$

$= 0.61836 + (3 - 0.8414) 2/3$

$\boxed{u(2/3) = 2.05738}$

→ put $x = 1$

$u(1) = \sin(1) + (3 - \sin 1) 1$

$= 0.8414 + (3 - 0.8414)$

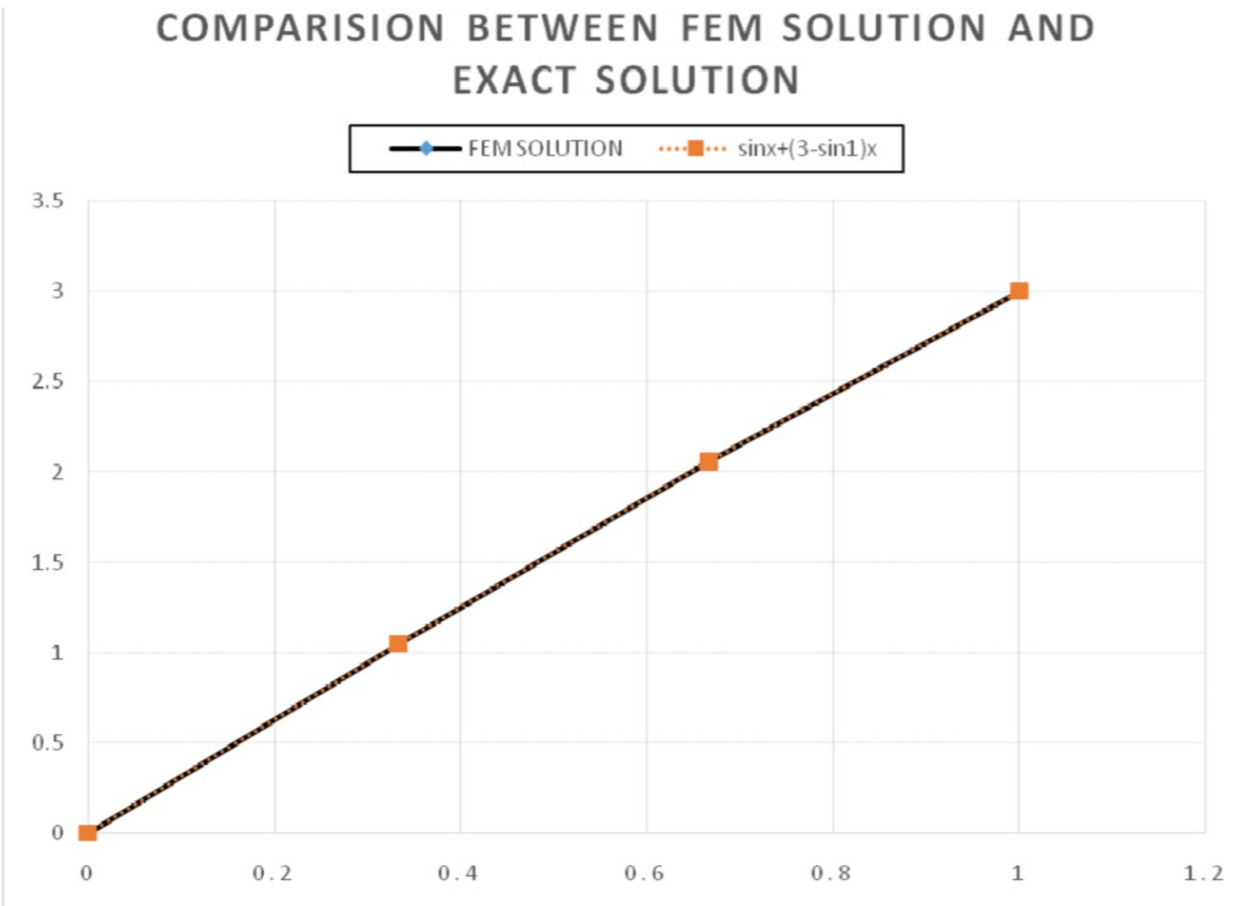
$\boxed{u(1) = 3}$

$\therefore$

$$\boxed{\begin{aligned} u(0) &= 0 \\ u(1/3) &= 1.04670 \\ u(2/3) &= 2.05738 \\ u(1) &= 3 \end{aligned}} \rightarrow \text{Exact Solution}$$

* Comparison :-

FEM Solution	Exact Solution	Relative Error
$u_1 = 0$	$u(0) = 0$	0
$u_2 = 1.0467$	$u(1/3) = 1.0467$	0
$u_3 = 2.0573$	$u(2/3) = 2.05738$	8×10^{-5}
$u_4 = 3$	$u(1) = 3$	0


COMPARISION BETWEEN FEM SOLUTION AND EXACT SOLUTION
FEM SOLUTION
sinx+(3-sin1)x
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2