Finite Element Method Rupalee Deepak Baldota

HOME WORK - 2

1] Solution : Plane Elasticity

Given : Young's modulus $E = 10$ GPa

Poisson Ratio $\nu = 0.2$

Thickness $t = 1$ m

Vertical Displacement $\delta = 10^{-2} = 0.01$ m

Body Force $\rho g = 10^{3}\ N/m^2$

Here Plane stress model is used to analyse plane elasticity ~~model~~ problem of prismatic bodies.

(i) Strong form :

The strong form is given by,

$$b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad \text{— (a)}$$

$$b_y + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad \text{— (b)}$$

Boundary Conditions :

B.C. are taken through displacement of given nodes in x and y direction.

From figure given, we get,

$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$ (fixed nodes 1, 2, 3)

$u_5 = 0$ (Symmetry condition)

$\delta = 10^{-5} = 0.01\,m = v_6$ (Given data)

(ii) Nodal Coordinates (x) and Connectivity matrix (T):

Nodes	x	y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Table ①: Nodal coordinates (x)

Element	Nodes		
	1	2	3
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

Table ②: T-Matrix (Connectivity Matrix)

* Description of Mesh:

From, fig, we have 4 elements in order to make the descritization easier. Local numbering is made such that in every element, the node in right angle vertex has a local number equal to 1.

To find the descritization of displacement field, we have,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The three nodes of triangular mesh defines linear displacement field which can be written as,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

After deriving the shape functions for 'u' alone, we get

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y) \quad \text{— (c)}$$

where,
$$a_i = x_j x_k - x_k x_j$$
$$b_i = y_j - y_k$$
$$c_i = x_k - x_j.$$

Stiffness Matrix is given by,

$$K^e = \iint_{A^e} B^T D B t \, dA \quad \text{—— (d)}$$

Equivalent nodal force vector,

$$f^e = f_\epsilon^e + f_\sigma^e + f_b^e + f_t^e \quad \text{— (e)}$$

where,
$$f_\epsilon^e = \iint_{A^e} B^T D \epsilon^0 t \, dA$$
$$f_\sigma^e = \iint_{A^e} B^T D \sigma^0 t \, dA$$
$$f_b^e = \iint_{A^e} N^T b t \, dA$$
$$f_t^e = \iint_{A^e} N^T t \, t \, dA$$

Eqn (d) can be written as,

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix} \quad \text{— (f)}$$

For plane stress problem,

$$\sigma = D\epsilon$$

where, D = constitutive matrix defined for given data.

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

where, $d_{11} = d_{22} = \dfrac{E}{(1-\nu^2)} = \dfrac{10}{(1-0.2^2)} = 10.417 \text{ GPa}$

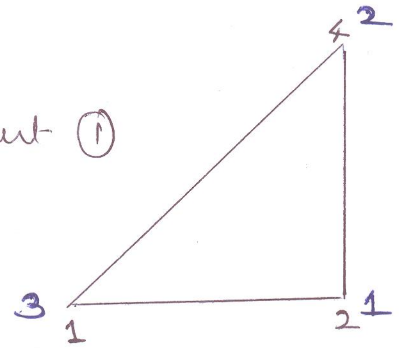
$d_{12} = d_{21} = \nu d_{11} = 0.2 \times (10.417) = 2.083 \text{ GPa}.$

$d_{33} = \dfrac{E}{2(1+\nu)} = \dfrac{10}{2(1+0.2)} = 4.167 \text{ GPa}.$

Computing Stiffness Matrix for Elements 1, 3 & 4:

From nodal coordinates & considering local numbering,

Element ①



1, 2, 4 → Global numbering
1, 2, 3 → Local numbering

From fig. ① : Element ①.

$(x_1, y_1)^1 = (-1.5, 0)$

$(x_2, y_2)^1 = (-1.5, 1.5)$

$(x_3, y_3)^1 = (-3, 0)$

But, $b_i = y_j - y_k$ & $c_i = x_k - x_j$

$b_1 = y_2 - y_3 = 1.5$; $c_1 = x_3 - x_2 = -1.5$

$b_2 = y_3 - y_1 = 0$; $c_2 = x_1 - x_3 = 1.5$

$b_3 = y_1 - y_2 = -1.5$; $c_3 = x_2 - x_1 = 0$

Now using eq^n (7), we have,

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

$$K_{11}^{①} = \frac{2}{9} \begin{bmatrix} 2.25 \times 10.417 + 2.25 \times 4.167 & -2.25 \times 2.083 - 2.25 \times 4.167 \\ -2.25 \times 2.083 + (-2.25) \times 4.167 & 2.25 \times 10.417 + 2.25 \times 4.167 \end{bmatrix}$$

$$= \frac{2 \times 2.25}{9} \begin{bmatrix} 10.417 + 4.167 & 2.083 - 4.167 \\ 2.083 - 4.167 & 10.417 + 4.167 \end{bmatrix}$$

$$K_{11}^{①} = \begin{bmatrix} 7.292 & -3.125 \\ -3.125 & 7.292 \end{bmatrix}$$

Similarly,

$$K_{12}^{①} = \frac{2 \times 2.25}{9} \begin{bmatrix} -4.167 & 2.083 \\ 4.167 & -10.417 \end{bmatrix} = \begin{bmatrix} -2.083 & 1.042 \\ 2.083 & -5.2085 \end{bmatrix} = K_{21}^e$$

$$K_{13}^{①} = \frac{2 \times 2.25}{9} \begin{bmatrix} -10.417 & 4.167 \\ 2.083 & -4.167 \end{bmatrix} = \begin{bmatrix} -5.2085 & 2.083 \\ 1.042 & -2.083 \end{bmatrix} = K_{31}^e$$

$$K_{22}^{①} = \frac{2 \times 2.25}{9} \begin{bmatrix} 4.167 & 0 \\ 0 & 10.417 \end{bmatrix} = \begin{bmatrix} 2.083 & 0 \\ 0 & 5.2085 \end{bmatrix}$$

$$K_{23}^{①} = \frac{2 \times 2.25}{9} \begin{bmatrix} 0 & -4.167 \\ -2.083 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.083 \\ -1.042 & 0 \end{bmatrix} = K_{32}^e$$

$$K_{33}^{①} = \frac{2 \times 2.25}{9} \begin{bmatrix} 10.417 & 0 \\ -2.083 & 4.167 \end{bmatrix} = \begin{bmatrix} 5.2085 & 0 \\ 0 & 2.083 \end{bmatrix}$$

Since Nodal and Local co-ordinates for elements ①, ③ & ④ are same, we have,

$K_{11}^{③} = K_{11}^{④} = K_{11}^{①}$; $K_{12}^{③} = K_{12}^{④} = K_{12}^{①}$; $K_{13}^{③} = K_{13}^{④} = K_{13}^{①}$

$K_{21}^{③} = K_{21}^{④} = K_{21}^{①}$; $K_{22}^{③} = K_{22}^{④} = K_{22}^{①}$; $K_{23}^{③} = K_{23}^{④} = K_{23}^{①}$

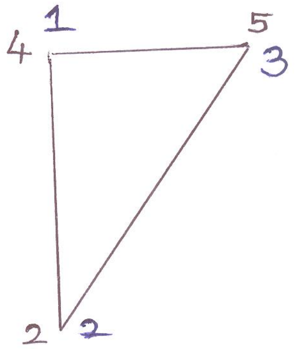
$K_{31}^{③} = K_{31}^{④} = K_{31}^{①}$; $K_{32}^{③} = K_{32}^{④} = K_{32}^{①}$; $K_{33}^{③} = K_{33}^{④} = K_{33}^{①}$

Therefore, element ①, ③ & ④ have same stiffness matrix

Computing Stiffness matrix for element (2) :

From Nodal co-ordinates, considering Local numbering, we have,

Element (2)



2, 4, 5 → Global numbering

1, 2, 3 → Local numbering

Fig. 2 : Element 2

From fig 2,

$(x_1, y_1)^2 = (-1.5, 1.5)$

$(x_2, y_2)^2 = (-1.5, 0)$

$(x_3, y_3)^2 = (0, 1.5)$

But, $b_i = y_j - y_k$ & $c_i = x_k - x_j$

$b_1 = y_2 - y_3 = -1.5$; $c_1 = x_3 - x_2 = 1.5$

$b_2 = y_3 - y_1 = 0$; $c_2 = x_1 - x_3 = -1.5$

$b_3 = y_1 - y_2 = 1.5$; $c_3 = x_2 - x_1 = 0$

Since values are same as that of element (1), but signs are opposite, we have the same stiffness matrix for element (2) as well,

Therefore, $K^1 = K^2 = K^3 = K^4$

* **Assembly of Stiffness Matrix :**

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} = K^2 = \begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}$$

$$K^3 = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{bmatrix} = K^4 = \begin{bmatrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{bmatrix}$$

The assembly of Matrix is as follows,

$$K = \begin{bmatrix} K_{33}^{(1)} & K_{13}^{(1)T} & 0 & K_{23} & 0 & 0 \\ & K_{11}^{(1)}+K_{22}^{(2)}+K_{33}^{(3)} & K_{13}^{(3)T} & K_{12}^{(1)}+K_{12}^{(2)T} & K_{23}^{(2)}+K_{23}^{(3)T} & 0 \\ & & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ & & & K_{22}^{(1)}+K_{11}^{(2)}+K_{33}^{(4)} & K_{13}^{(2)}+K_{13}^{(4)T} & K_{23}^{(4)T} \\ & & & & K_{11}^{(4)}+K_{22}^{(3)}+K_{33}^{(2)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

We know that,

$$a_i^e = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \Rightarrow a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

For the given problem, displacement matrix is given by,

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

And Nodal force vector,

$$f = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix}$$

Therefore $Ka = f$

* The given problem has 12 degrees of freedom, where 9 degrees it is constrained.

Computing Nodal Displacement :

From Boundary conditions, we have,

$$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0 ;$$

Hence, $a_1 = a_2 = a_3 = 0$

Hence, we need to consider only 4, 5 & 6 rows only from global stiffness matrix, and also,

$$u_5 = u_6 = 0$$

Therefore,

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & K_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ u_5 = 0 \\ v_5 \\ u_6 = 0 \\ v_6 = \delta = 10^{-2} \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix} \quad \text{— (f)}$$

Now, substituting value of k_{ij}^{e} for the above matrix,

$$K = \begin{bmatrix} -14.5835 & -3.125 & -10.417 & 3.125 & 0 & 1.042 \\ -3.125 & 14.5835 & 4.166 & -4.166 & -1.042 & 0 \\ -10.417 & 4.166 & 14.5835 & -3.125 & -2.083 & 1.042 \\ 3.125 & -4.166 & -3.125 & 14.5835 & 2.083 & -5.2085 \\ 0 & -1.042 & -2.083 & 2.083 & 2.083 & 0 \\ 1.042 & 0 & 1.042 & -5.2085 & 0 & 5.2085 \end{bmatrix} \text{GN/m}$$

We have from data that the whole domain deforms because of self weight with gravity acting in the direction y-axis. Therefore, only body forces are significant & no surface loads.

Body forces for the equivalent nodal force is given by,

$$f_{b_i} = \left(\frac{At}{3}\right)^{e} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \longrightarrow (g)$$

but $b_x = 0$; $b_y = -\rho g = -10^{3}$

$$f_{bi} = \left(\frac{2.25}{6}\right)\begin{bmatrix} 0 \\ -10^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \end{bmatrix} N$$

Substituting f_{bi}^e & K in eq^n (f), we get by simplification

$$\begin{bmatrix} 14.5835 & -3.125 & 3.125 & 1.042 \\ -3.125 & 14.5835 & -4.166 & 0 \\ 3.125 & -4.166 & 14.5835 & -5.2085 \\ 1.042 & 0 & -5.2085 & 5.2085 \end{bmatrix} \times 10^9 \begin{bmatrix} u_4 \\ V_4 \\ V_5 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

When we solve the above system of linear equations we get the nodal displacement values as;

$$\boxed{\begin{aligned} u_4 &= -1.29 \times 10^{-4} \\ V_4 &= -1.13 \times 10^{-3} \\ V_5 &= -3.87 \times 10^{-3} \end{aligned}}$$