# 1.

We are going to analyze this problem using a plane stress model.

The deformation in 2 dimensions can be described with:

$$\sigma = D\,\varepsilon$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E(1-v)}{(1+v)(1-2v)} & \frac{Ev}{(1+v)(1-2v)} & 0 \\ \frac{Ev}{(1+v)(1-2v)} & \frac{E(1-v)}{(1+v)(1-2v)} & 0 \\ 0 & 0 & \frac{E}{2(1+v)} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

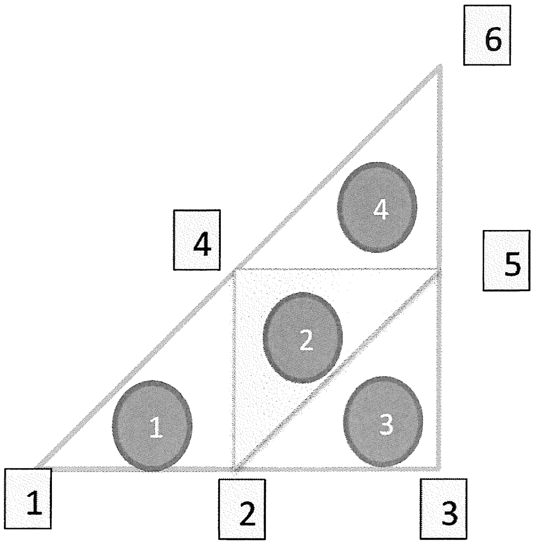
$$\varepsilon_x = \frac{1+v}{E}\left((1-v)\,\sigma_x - v\sigma_y\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1+v}{E}\left((1-v)\,\sigma_y - v\sigma_x\right) \qquad \gamma_{yz} = 0$$

$$\varepsilon_z = 0 \qquad \gamma_{xz} = 0$$

$$\sigma_z = v(\sigma_x + \sigma_y)$$

where: σ is the stress and ε the strain.D is the constitutive matrix.
E= 10 GPa and ν=0.2




The numbers in the yellow boxes are the 6 nodes and we have 4 triangle elements (numbered in blue circles) inside our domain (the big triangle).

The **strong form** in this kind of linear elastic problem is:

**div(σ) + b = 0** in our domain problem

b is the volumetric force vector

And the **boundary conditions:**

$u_1 = (0,0)$
$u_2 = (0,0)$
$u_3 = (0,0)$
$u_4 = (0, u_{4y})$
$u_5 = (0, u_{5y})$
$u_6 = (0, \delta)$

$u_i$=$(u_x, u_y)$

The deformation affect just in y coordinates and in the 4, 5 and 6 nodes. Displacement in 6 node is given ($\delta$).

## 2.

**Global numbering:**

We choose our Origin of **coordinates** in node 3, so:

| node | x | y |
|---|---|---|
| 1 | -3 | 0 |
| 2 | -1.5 | 0 |
| 3 | 0 | 0 |
| 4 | 1,5 | 1,5 |
| 5 | 0 | 1,5 |
| 6 | 0 | 3 |

$$K^{(e)} = \frac{k}{4A^{(e)}} \begin{bmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{bmatrix}$$

$$b_i = y_j^{(e)} - y_k^{(e)}$$

$$c_i = x_k^{(e)} - x_j^{(e)}$$

$$K^{(1)} = K^{(3)} = K^{(4)} = \left(\frac{k}{4A}\right) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$K^{(2)} = \left(\frac{k}{4A}\right) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$K = \begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 & 0 \\ & K_{22}^{(1)} + K_{11}^{(3)} + K_{11}^{(4)} & K_{23}^{(1)} + K_{13}^{(3)} & K_{12}^{(1)} & K_{13}^{(1)} + K_{12}^{(3)} & 0 \\ & & K_{33}^{(1)} + K_{33}^{(3)} + K_{11}^{(4)} & 0 & K_{32}^{(3)} + K_{11}^{(4)} & K_{13}^{(4)} \\ & & & K_{22}^{(2)} & K_{23}^{(2)} & 0 \\ & \text{sym} & & & K_{22}^{(2)} + K_{22}^{(3)} + K_{22}^{(4)} & K_{23}^{(4)} \\ & & & & & K_{33}^{(4)} \end{pmatrix}$$

$Ka = f$

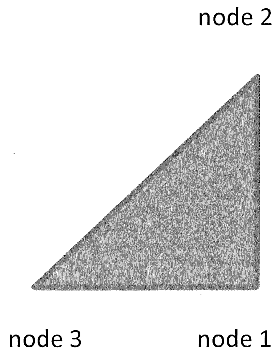
↳ obtain a

Connectivity matrix $T = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{pmatrix}$

| element | nodes | | |
|---|---|---|---|
| 1 | 2 | 4 | 1 |
| 2 | 4 | 2 | 5 |
| 3 | 3 | 5 | 2 |
| 4 | 5 | 6 | 4 |

**Local numbering:**

In every element, the node in the right angle vertex is the node 1.
So in **elements 1, 3 and 4**:




| node | x | y |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | +1 |
| 3 | -1 | 0 |

In natural coordinates:

(x, y) ---> (α, β)

The shape functions will be:

$N_1 = 1 - \beta + \alpha$

$N_2 = \beta$

$N_3 = \alpha$

$\Phi = a + b\alpha + c\beta$

$\Phi_1 = a + b\alpha_1 + c\beta_1$

$\Phi_2 = a + b\alpha_2 + c\beta_2$

$\Phi_3 = a + b\alpha_3 + c\beta_3$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$a = \Phi_1$

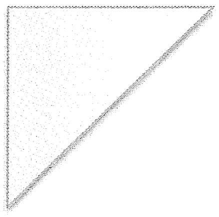
$b = (\Phi_1 + \Phi_3)$

$c = (\Phi_2 - \Phi_1)$

$\Phi = \Phi_1 + \alpha(\Phi_1 + \Phi_3) + \beta(\Phi_2 - \Phi_1)$

$\Phi = (1 - \beta + \alpha)\,\Phi_1 + \beta\Phi_2 + \alpha\Phi_3$

And in **element 2**:




| node | x | y |
|---|---|---|
| 1 | 0 | 0 |
| 2 | -1 | 0 |
| 3 | -1 | -1 |

$N_1 = \alpha$
$N_2 = 1 - \alpha + \beta$
$N_3 = \beta$

$\Phi = a + b\alpha + c\beta$

$\Phi_1 = a + b\alpha_1 + c\beta_1$
$\Phi_2 = a + b\alpha_2 + c\beta_2$
$\Phi_3 = a + b\alpha_3 + c\beta_3$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

**a =** $\boldsymbol{\Phi_2}$
**b =** $-\boldsymbol{\Phi_2} + \boldsymbol{\Phi_1}$
**c =** $\boldsymbol{\Phi_3} + \boldsymbol{\Phi_2}$

$\Phi = \Phi_2 + \alpha(-\Phi_2 + \Phi_1) + \beta(\Phi_2 + \Phi_3)$
$\Phi = \alpha\,\Phi_1 + (1 - \alpha + \beta)\Phi_2 + \beta\Phi_3$

# 3.

The global equilibrium is

$$K\,a = f$$

And we have to calculate it for each element:

$$K^{(e)} a^{(e)} = f^{(e)}$$

For a 3-noded triangular element:

$$\mathbf{K}_{ij}^{(e)} = \left(\frac{t}{4A}\right)^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

And

$$\mathbf{f}_{b_i} = \frac{(At)^{(e)}}{3} \begin{Bmatrix} bx \\ by \end{Bmatrix}$$

For element 1,3 and 4 we have:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} 1{,}1\times10^{10} & 0{,}3\times10^{10} & 0 \\ 0{,}3\times10^{10} & 1{,}1\times10^{10} & 0 \\ 0 & 0 & 0{,}4\times10^{10} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

$$\varepsilon_x = 1{,}2\times10^{-10}\,(0{,}8\sigma_x - 0{,}2\sigma_y)$$
$$\varepsilon_y = 1{,}2\times10^{-10}\,(0{,}8\sigma_y - 0{,}2\sigma_x)$$
$$\varepsilon_z = 0$$

$$\sigma_x = 1,1 \times 10^{10}\, \varepsilon_x + 0,3 \times 10^{10}\, \varepsilon_y$$

$$\sigma_y = 0,3 \times 10^{10}\, \varepsilon_x + 1,1 \times 10^{10}\, \varepsilon_y$$

$$\tau_{xy} = \gamma_{xy}\,(0,4 \times 10^{10})$$

$$\frac{\tau_{xy}}{G} = \gamma_{xy}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\sigma_z = 0,2(\sigma_x + \sigma_y)$$

coordinates local

| | nodes |
|---|---|
| $i$ | 1 |
| $j$ | 2 |
| $k$ | 3 |

$x_1 = 0 \quad y_1 = 0$

$x_2 = 1 \quad y_2 = 0$

$x_3 = -1 \quad y_3 = 0$

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

$$a_1 = x_2 y_3 - x_3 y_2 = 0 - 0 = 0$$

$$b_1 = y_2 - y_3 = 0$$

$$c_1 = x_3 - x_2 = -1 - 1 = -2$$

$$A^{(e)} = \frac{1}{2} b h = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{8}$$

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y)$$

$$N_1 = \frac{4}{9}(-2y) = \frac{-8}{9} y$$

$$a_2 = x_1 y_3 - x_3 y_1 = 0 - (-1)\,0 = 0$$
$$b_2 = y_1 - y_3 = 0$$
$$c_2 = x_3 - x_1 = -1 - 0 = -1$$

$$N_2 = \frac{4}{9}(-y) = \frac{-4}{9}y$$

$$a_3 = x_1 y_2 - x_2 y_1 = 0 - 0 = 0$$
$$b_3 = y_1 - y_2 = 0$$
$$c_3 = x_2 - x_1 = 1$$

$$N_3 = \frac{4}{9}(y) = \frac{4y}{9}$$

$$B_i = \begin{pmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{pmatrix}$$

$$B = \begin{array}{cc} & \begin{array}{cccccc} B_1 & & B_2 & & B_3 & \end{array} \\ & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{9} & 0 & -\frac{4}{9} & 0 & \frac{4}{9} \\ -\frac{8}{9} & 0 & \frac{4}{9} & 0 & \frac{4}{9} & 0 \end{pmatrix} \end{array}$$

$$K_{ij}^{(1)} = \left(\frac{t}{4A}\right)^{(1)} \begin{pmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{21} \end{pmatrix}$$

(See page 7)

$a_1 = 0$ $\quad a_2 = 0$ $\quad a_3 = 0$

$b_1 = 0$ $\quad b_2 = 0$ $\quad b_3 = 0$

$c_1 = -2$ $\quad c_2 = -1$ $\quad c_3 = 1$

$d_{11} = 1,1 \times 10^{10}$ $\quad d_{12} = 0,3 \times 10^{10}$ $\quad d_{13} = 0$

$d_{21} = 0.3 \times 10^{10}$ $\quad d_{22} = 1,1 \times 10^{10}$ $\quad d_{23} = 0$

$d_{31} = 0$ $\quad d_{32} = 0$ $\quad d_{33} = 0,4 \times 10^{10}$

(also at the end of page 7)

$$K_{ij}^{(1)} = K_{ij}^{(3)} = K_{ij}^{(4)} = K_{ij}$$

$$K_{11} = \left(\frac{t}{4A}\right) \begin{pmatrix} b_1 b_1 d_{11} + c_1 c_1 d_{33} & b_1 c_1 d_{12} + b_1 c_1 d_{33} \\ c_1 b_1 d_{21} + b_1 c_1 d_{33} & b_1 b_1 d_{33} + c_1 c_1 d_{22} \end{pmatrix}$$

$$K_{11} = \left(\frac{t}{4A}\right) \begin{pmatrix} 0 - 2(-1)\,0,4 \times 10^{10} & 0 + 0 \\ (-2)\,0 + 0 & 0 + (-2)(-2) \times 1,1 \times 10^{10} \end{pmatrix}$$

$$K_{11} = \left(\frac{t}{4A}\right) 10^{10} \begin{pmatrix} 0,8 & 0 \\ 0 & 4,4 \end{pmatrix}$$

$$K_{12} = \left(\frac{t}{4A}\right)\begin{pmatrix} b_1 b_2 d_{11} + c_1 c_2 d_{33} & b_1 c_2 d_{12} + b_2 c_1 d_{33} \\ c_1 b_2 d_{21} + b_1 c_2 d_{33} & b_1 b_2 d_{33} + c_1 c_2 d_{22} \end{pmatrix}$$

$$K_{11} = \left(\frac{t}{4A}\right)\begin{pmatrix} 0 + (-2)(-1)\,0{,}4\times 10^{10} & 0 + 0 \\ 0 + 0 & 0 + (-2)(-1)\,1{,}1\times 10^{10} \end{pmatrix}$$

$$K_{11} = \left(\frac{t}{4A}\right)10^{10}\begin{pmatrix} 0{,}8 & 0 \\ 0 & 2{,}2 \end{pmatrix}$$

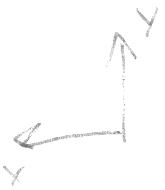
$$K_{21} = K_{12}$$

$$K_{13} = K_{31} = \left(\frac{t}{4A}\right)\begin{pmatrix} 0 + (1)(-2)\times 10^{10}\times 0{,}4 & 0 + 0 \\ 0 + 0 & 0 + (-2)(+1)10^{10}\times 1{,}1 \end{pmatrix}$$

$$K_{13} = K_{31} = \left(\frac{t}{4A}\right)10^{10}\begin{pmatrix} -0{,}8 & 0 \\ 0 & -2{,}2 \end{pmatrix}$$

$$K_{22} = \left(\frac{t}{4A}\right)10^{10}\begin{pmatrix} 0{,}4 & 0 \\ 0 & 1{,}1 \end{pmatrix}$$

$$K_{33} = \left(\frac{t}{4A}\right)10^{10}\begin{pmatrix} 0{,}4 & 0 \\ 0 & 1{,}1 \end{pmatrix}$$




$b_x = 0$

$b_y = -b$

$$f_b = \frac{At}{3}\begin{pmatrix} 0 \\ -b \end{pmatrix}$$

For element 2

| coordinate | nodes | local coordinate |
|---|---|---|
| i | 1 | $x_1 = 0 \quad y_1 = 0$ |
| j | 2 | $x_2 = -1 \quad y_2 = 0$ |
| k | 3 | $x_3 = -1 \quad y_3 = -1$ |

$$\left.\begin{aligned} a_1 &= 1 \\ b_1 &= 1 \\ c_1 &= 0 \end{aligned}\right\} N_1 = \frac{4}{9}$$

$$\left.\begin{aligned} a_2 &= 0 \\ b_2 &= -1 \\ c_2 &= 1 \end{aligned}\right\} N_2 = \frac{4}{9}(-x+y)$$

$$F_b = \frac{At}{3}\begin{pmatrix} 0 \\ -b \end{pmatrix}$$

$$\left.\begin{aligned} a_3 &= 0 \\ b_3 &= 0 \\ c_3 &= -1 \end{aligned}\right\} N_3 = \frac{4}{9}(-y)$$

$$B = \begin{pmatrix} 0 & 0 & -\frac{4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & 0 & -\frac{4}{9} \\ 0 & 0 & \frac{4}{9} & -\frac{4}{9} & -\frac{4}{9} & 0 \end{pmatrix}$$

We must assemble the elments

And we obtain all the $k_i$ of those element.

Finaly we obtain the $K_{global} = \sum_{i=1}^{4} k_i$

Also the same with $f_{global}$

$$k = af$$

In this probleme we have 3 degrees of freedom:

$u_4$ (horizontal displacement in node 4)

$V_4$ (vertical displacement of node 4)

$V_5$ (vertical displacement of node 5)

So system will be reduced to only 3 equations

$$K a = F$$

$$\begin{pmatrix} k_{33}^{(1)} + k_{11}^{(2)} + k_{55}^{(4)} & k_{34}^{(1)} + k_{12}^{(2)} + k_{56}^{(4)} & k_{16}^{(2)} + k_{52}^{(4)} \\ k_{43}^{(1)} + k_{21}^{(2)} + k_{65}^{(4)} & k_{44}^{(1)} + k_{22}^{(2)} + k_{66}^{(4)} & k_{26}^{(2)} + k_{62}^{(4)} \\ k_{61}^{(2)} + k_{25}^{(4)} & k_{62}^{(2)} + k_{26}^{(4)} & k_{66}^{(2)} + k_{44}^{(3)} + k_{22}^{(4)} \end{pmatrix} \begin{pmatrix} u_7 \\ u_8 \\ u_{10} \end{pmatrix} =$$

$$= \begin{pmatrix} f_3^{(1)} + f_1^{(2)} + f_5^{(4)} \\ f_4^{(1)} + f_2^{(2)} + f_6^{(4)} \\ f_6^{(2)} + f_4^{(3)} + f_2^{(4)} \end{pmatrix} - \begin{pmatrix} k_{7,12}\delta \\ k_{8,12}\delta \\ k_{10,12}\delta \end{pmatrix}$$

We compute

$$K^{(1)} = K^{(2)} = K^{(4)}$$

and

$$K^{(3)}$$

and obtain the global stiffness matrices:

$$K = 10^{10} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{array} \begin{pmatrix} 0{,}5 & 0 & -0{,}5 & 0{,}1 & 0 & 0 & 0 & -0{,}1 & 0 & 0 & 0 & 0 \\ 0 & 0{,}2 & 0{,}2 & -0{,}2 & 0 & 0 & -0{,}2 & 0 & 0 & 0 & 0 & 0 \\ -0{,}5 & 0{,}2 & 1{,}5 & -0{,}3 & -0{,}5 & 0{,}1 & -0{,}4 & 0{,}3 & 0 & -0{,}3 & 0 & 0 \\ 0{,}1 & -0{,}2 & -0{,}3 & 1{,}5 & 0{,}2 & -0{,}2 & 0{,}3 & -1 & -0{,}3 & 0 & 0 & 0 \\ 0 & 0 & -0{,}5 & 0{,}2 & 0{,}8 & -0{,}3 & 0 & 0 & -0{,}2 & 0{,}1 & 0 & 0 \\ 0 & 0 & 0{,}1 & -0{,}2 & -0{,}3 & 0{,}7 & 0 & 0 & 0{,}2 & -0{,}5 & 0 & 0 \\ 0 & -0{,}2 & -0{,}4 & 0{,}3 & 0 & 0 & 1{,}5 & -0{,}3 & -1 & 0{,}3 & 0 & -0{,}1 \\ -0{,}1 & 0 & 0{,}3 & -1 & 0 & 0 & -0{,}3 & 1{,}5 & 0{,}3 & -0{,}4 & -0{,}2 & 0 \\ 0 & 0 & 0 & -0{,}3 & -0{,}1 & 0{,}2 & -1 & 0{,}3 & 1{,}5 & -0{,}3 & -0{,}2 & 0{,}1 \\ 0 & 6 & -0{,}3 & 0 & 0{,}1 & -0{,}5 & 0{,}3 & -0{,}4 & -0{,}3 & 1{,}5 & 0{,}2 & -0{,}5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0{,}2 & -0{,}1 & 0 & 0{,}2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0{,}1 & 0 & 0{,}1 & -0{,}2 & 0 & 0{,}5 \end{pmatrix}$$

forget yellow line

$$F_{global} = \sum f_i = \begin{pmatrix} 0 \\ -375 \\ 0 \\ +1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1225 \\ 0 \\ -375 \end{pmatrix}$$

$u_7 = u_4$

$u_8 = v_4$

$u_{10} = v_5$

global local

$$10^{10}\begin{pmatrix} 1,5 & -1 & 0,3 \\ 0,3 & 0,3 & -0,4 \\ -0,1 & -0,3 & 1,5 \end{pmatrix}\begin{pmatrix} u_7 \\ u_8 \\ u_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1225 \end{pmatrix} - \begin{pmatrix} -0,1\times 10^{-3} \\ 0 \\ -0,5 \end{pmatrix}$$

$$10^{10}(1,5u_7 - u_8 + 0,3u_{10}) = -0,1\times 10^{-3}$$

$$10^{10}(0,3u_7 + 0,3u_8 - 0,4u_{10}) = 0$$

$$10^{10}(-0,1u_7 - 0,3u_8 + 1,5u_{10}) = -1225 + 0,5$$

$u_1 = 0$

$u_2 = 0$

$u_3 = 0$

$u_4 = 0$

$u_5 = 0$

$u_6 = 0$

$u_7 = (u_{4\,local}) = -10^{-4}$

$u_8 = (V_{4\,local}) = -10^{-3}$

$u_9 = 0$

$u_{10} = (V_{5\,local}) = -4\times 10^{-4}$

$u_{11} = 0$

$u_{12} = 10^{-2}$

So,

$$\varepsilon = Bu$$

We assemble $B^{(1)(3)(4)}$ and $B^{(2)}$ to get $B$

$$u_{glosp} = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -10^{-4} \;\; -10^{-3} \;\; 0 \;\; -4\cdot 10^{-4} \;\; 0 \;\; 10^{-2})^T$$

And we get $\varepsilon$

element 1

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{9} & 0 & \frac{-4}{9} & 0 & \frac{4}{9} \\ \frac{-8}{9} & 0 & \frac{-4}{9} & 0 & \frac{4}{9} & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \\ u_1 \\ v_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -10^{-4} \\ -10^{-3} \\ 0 \\ 0 \end{pmatrix}$$

$$\varepsilon_x = 0$$

$$\varepsilon_y = \frac{4}{9} 10^{-3}$$

$$\gamma_{xy} = \frac{4}{9} 10^{-4}$$

element 2

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{-4}{9} \\ 0 & 0 & \frac{4}{9} & \frac{-4}{9} & \frac{-4}{9} & 0 \end{pmatrix} \begin{pmatrix} u_4 = 10^{-4} \\ v_4 = -10^{-3} \\ u_2 = 0 \\ v_2 = 0 \\ u_5 = 0 \\ v_5 = -4\times 10^{-4} \end{pmatrix}$$

$$\varepsilon_x = 0$$

$$\varepsilon_y = +\frac{16}{9}\times 10^{-4}$$

$$\gamma_{xy} = 0$$

$\boxed{\text{element } 3}$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \frac{4}{9} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ -2 & 0 & -1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} u_3 = 0 \\ v_3 = 0 \\ u_5 = 0 \\ v_5 = -4\times10^{-4} \\ u_2 = 0 \\ v_2 = 0 \end{pmatrix}$$

$$\varepsilon_x = 0$$

$$\varepsilon_y = \frac{4}{9}\left(4\times10^{-4}\right) = \frac{16}{9}\times10^{-4}$$

$$\gamma_{xy} = 0$$

$\boxed{\text{element } 4}$

$$u = \begin{pmatrix} u_5 = 0 \\ v_5 = -4\times10^{-4} \\ u_6 = 0 \\ v_6 = 10^{-2} \\ u_4 = -10^{-4} \\ v_4 = -10^{-3} \end{pmatrix}$$

$$\varepsilon_x = 0$$

$$\varepsilon_y = \frac{4}{9}\left(-2\times(-4\times10^{-4}) - 10^{-2} - 10^{-3}\right) = \frac{-4}{9}(0{,}0102)$$

$$\gamma_{xy} = \frac{4}{9}\left(-2\times10^{-4}\right) = \frac{-8}{9}\times10^{-4}$$

$$\sigma = D\varepsilon$$

For elements 1, 3, and 4 (see page 7)

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = \begin{pmatrix} 1,1\times10^{10} & 0,3\times10^{10} & 0 \\ 0,3\times10^{10} & 1,1\times10^{10} & 0 \\ 0 & 0 & 0,4\times10^{10} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

Element 1.

$$\sigma_x = 1,1\times10^{10}\times0 + 0,3\times10^{10}\left(\frac{4}{9}10^{-3}\right) = 6,75\times10^{6}$$

$$\sigma_y = 0,3\times0 + 1,1\times10^{10}\left(\frac{4}{9}10^{-3}\right) = 4,9\times10^{6}$$

$$\tau_{xy} = \gamma_{xy}(0,4\times10^{10}) = 0,2\times10$$

Element 2

$$\sigma_x = 0 + \frac{16}{9}\times10^{-4}\times0,3\times10^{10} = -0,5\times10^{6}$$

$$\sigma_y = 0 + 1,1\times10^{10}\times\frac{16}{9}\times10^{-4} = 2\times10^{6}$$

$$\tau_{xy} = 0$$

$\boxed{\text{Element 3}}$

$$\sigma_x = \frac{16}{9} \times 10^{-4} \times 0{,}3 \times 10^{10} = 2{,}7 \times 10^{7}$$

$$\sigma_y = 1{,}1 \times 10^{10} \times \frac{16}{9} \times 10^{-4} = 2{,}9 \times 10^{7}$$

$$\tau_{xy} = 0$$

$\boxed{\text{Element 4}}$

$$\sigma_x = \frac{-4}{9}(0{,}0102) \times 3 \times 10^{10} = -0{,}136 \times 10^{9}$$

$$\sigma_y = 1{,}1 \times 10^{10} \times \frac{-4}{9}(0{,}0102) = -0{,}05 \times 10^{9}$$

$$\tau_{xy} = -0{,}36 \times 10^{6}$$

And the reaction force will be calculated

$$R = ku - f$$

k in page 14

$$f = (0 \; -375 \; 0 \; -1125 \; 0 \; -375 \; 0 \; -1125 \; 0 \; -1125 \; 0 \; -375)^T$$

$$u = (0 \; 0 \; 0 \; 0 \; 0 \; 0 \; -10^{-4} \; -10^{-3} \; 0 \; -4 \times 10^{-4} \; 0 \; 10^{-2})^T$$

$$R = \begin{pmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ R_{3x} \\ R_{3y} \\ R_{4x} \\ R_{4y} \\ R_{5x} \\ R_{5y} \\ R_{6x} \\ R_{6y} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 9 \\ 11 \\ -4 \\ 2 \\ 0 \\ 0 \\ -0{,}5 \\ 0 \\ -5{,}7 \\ -3 \end{pmatrix} \times 10^{6}$$

Please, forgive me for the presentation. I am working and I do not have much time.