FINITE ELEMENT

Homework 1

Christian Rossi

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Consider the following differential equation

$$\frac{d^2u}{dx^2} = -Q \quad in \quad (0,1)$$

With follow boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \ldots, n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equation to be solved.
3. Compute the FE approximation u^h for $n = 3$, $Q(x) = \sin x$ and $\alpha = 3$. Compute it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$.

The governing equation of the problem is:

$$\frac{d^2u}{dx^2} = -Q \quad in \quad (0,1) \tag{1.1}$$

With the following Dirichlet boundaries condition:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

The differential equation, described in strong form, can be transformed in an equivalent integral expression by multiplying it by an arbitrary weighting function, and integrating over the domain. Thus

$$\int_0^l W(x) \cdot \left(\frac{d^2u}{dx^2} + Q\right) dx = 0 \tag{1.2}$$

Where W(x) is the weighting function and the integral statements Eq.(1.2) is equivalent to the differential equation Eq.(1.1).

The unknown function u can be approximated by a linear combination of function as:

$$u \approx u^h = \sum_{j=1}^{n} N_j a_j \tag{1.3}$$

Where *n* is the number of the nodes, $N_j(x)$ is the *shape function* and a_j is unknown parameter. This concept allows us to approximate a continuous function using a discrete model. The continuous function is divided into finite elements and the discrete model is composed of interpolation polynomials. The behavior of each element is described using the shape function between its end points. The shape function is written for each node of each element and has the property that its magnitude is 1 at the node and 0 elsewhere. The shape function is characterized for be continuous over the domain and satisfied the boundary condition.

Substituting the approximate function u^h into the integral form (1.2) we obtain an approximation of the integral form called weight residual expression.

$$\int_0^l W(x) \cdot \left(\frac{d^2 u^h}{dx^2} + Q \right) dx = 0 \tag{1.4}$$

Using the equation (1.3) we can rewrite the above equation in a discrete form, thus

$$\int_0^l W_i(x) \cdot \frac{d^2}{dx^2} \left(\sum_{j=0}^{n} N_j a_j \right) dx + \int_0^l W_i(x) \cdot Q \, dx = 0 \tag{1.5}$$

Integrating the first term of the Eq.(1.4) by integration by parts it gives an equation with only first order derivatives.

$$\int_0^l W_i(x) \cdot \frac{d^2 u^h}{dx^2} dx = W_i(x) \cdot \frac{du^h}{dx} \bigg|_0^l - \int_0^l \frac{du^h}{dx} \cdot \frac{dW_i(x)}{dx} dx \tag{1.6}$$

Using Eq.(1.6) the weak form of the problem reads

$$\int_0^l \frac{du^h}{dx} \cdot \frac{dW_i(x)}{dx} dx = W_i(x) \cdot \frac{du^h}{dx} \bigg|_0^l + \int_0^l W_i(x) \, Q \, dx \tag{1.7}$$

Using the Galerkin method, which is distinguished for its accuracy and simplicity, we choose the follow expression:

$$W_i = N_i$$

And we call

$$q = \frac{du_i^h}{dx}$$

The weak form equation using Galerkin methods reads,

$$\int_0^l \frac{dN_i}{dx} \cdot \frac{dN_j(x)}{dx} a_j dx = N_i(x) \cdot \mathrm{q} \big|_0^l + \int_0^l N_i(x) \, Q \, dx \tag{1.8}$$

And it gives us the following system of *n* equations and *n* unknowns by giving values from *i= 0 to n.*

For i=0

$$\int_0^l \frac{dN_0(x)}{dx} \cdot \left(\frac{dN_0(x)}{dx} a_0 + \frac{dN_1(x)}{dx} a_1 + \cdots + \frac{dN_n(x)}{dx} a_n \right) dx = N_0(x) \cdot \mathrm{q} \Big|_0^l + \int_0^l N_0(x)\, Q\, dx$$

For i=1

$$\int_0^l \frac{dN_1(x)}{dx} \cdot \left(\frac{dN_0(x)}{dx} a_0 + \frac{dN_1(x)}{dx} a_1 + \cdots + \frac{dN_n(x)}{dx} a_n \right) dx = N_1(x) \cdot \mathrm{q} \Big|_0^l + \int_0^l N_1(x)\, Q\, dx$$

For i=n

$$\int_0^l \frac{dN_n(x)}{dx} \cdot \left(\frac{dN_0(x)}{dx} a_1 + \frac{dN_1(x)}{dx} a_2 + \cdots + \frac{dN_n(x)}{dx} a_n \right) dx = N_n(x) \cdot \mathrm{q} \Big|_0^l + \int_0^l N_n(x)\, Q dx$$

The above equations for any value of *n* can be expressed in a compact form as:

$$\int_0^l \frac{dN_i(x)}{dx} \cdot \frac{dN_j(x)}{dx} a_j dx = N_i(x) \cdot \mathrm{q} \Big|_0^l + \int_0^l N_i(x)\, Q dx \qquad (1.10)$$

Above equation can be written in matrix form as

$$Ka = f$$

Where:

$$Kij = \int_0^l \frac{dN_i(x)}{dx} \cdot \frac{dN_j(x)}{dx} dx$$

$$f_i = N_i(x) \cdot \mathrm{q}\Big|_0^l + \int_0^l N_i(x)\, Q\, dx$$

Where K is called *stiffness matrix* and it is characterized to be a symmetrical matrix. The vector *a* contains the *n* unknowns parameters and *f* is the external forces vector.

The dimension of the *stiffness matrix* is [n x n], the unknown vector *a* is [n x 1] and the external forces vector *f* is [n x 1].

$$\begin{pmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & K_{ij} & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_i \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_i \\ f_n \end{pmatrix}$$

Before to compute the FE approximation for n = 3, Q(x) = sin x and = 3 , the all domain has to be discretized. The domain goes from 0 to 1 and it will be divided by the nodes $x_i = ih$ for $= 0,1\ldots,n$ and $h=1/n$.

So the nodes 1D x coordinates are:

$$i_0 = 0$$

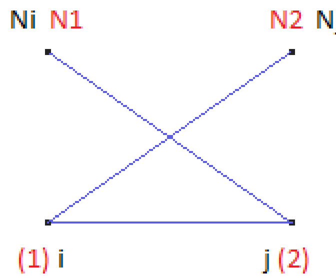
$$i_1 = 1/3$$

$$i_2 = 2/3$$

$$i_3 = 1$$

And the domain discretized in a global numbering is:

Considering just a single element of the domain, it can be described using a local numeration (in red) as:



So using the local numeration for each element of the whole domain, for each value of *i*, we obtain the following equations:

For i = 0

$$\int_0^{l/3} \frac{dN_0^{(1)}}{dx} \cdot \left[\frac{dN_0^{(1)}}{dx} \cdot a_0 + \frac{dN_1^{(1)}}{dx} \cdot a_1\right] \cdot dx = N_0^{(1)} \cdot \mathrm{q}\Big|_0^{l/3} + \int_0^{l/3} \frac{dN_0^{(1)}}{dx} Q \cdot dx$$

For i = 1

$$\int_0^{l/3} \frac{dN_1^{(1)}}{dx} \cdot \left[\frac{dN_0^{(1)}}{dx} \cdot a_0 + \frac{dN_1^{(1)}}{dx} \cdot a_1\right] \cdot dx + \int_{l/3}^{2l/3} \frac{dN_0^{(2)}}{dx} \cdot \left[\frac{dN_0^{(2)}}{dx} \cdot a_0 + \frac{dN_1^{(2)}}{dx} \cdot a_1\right] \cdot dx$$

$$= N_1^{(1)} \cdot \mathrm{q}\Big|_0^{l/3} + N_0^{(2)} \cdot \mathrm{q}\Big|_{l/3}^{2l/3} + \int_0^{l/3} \frac{dN_1^{(1)}}{dx} Q \cdot dx + \int_{l/3}^{2l/3} \frac{dN_0^{(2)}}{dx} Q \cdot dx$$

For i = 2

$$\int_{l/3}^{2l/3} \frac{dN_1^{(2)}}{dx} \cdot \left[\frac{dN_0^{(2)}}{dx} \cdot a_1 + \frac{dN_1^{(2)}}{dx} \cdot a_2\right] \cdot dx + \int_{2l/3}^{l} \frac{dN_0^{(3)}}{dx} \cdot \left[\frac{dN_0^{(3)}}{dx} \cdot a_1 + \frac{dN_1^{(3)}}{dx} \cdot a_2\right] \cdot dx$$

$$= N_1^{(2)} \cdot \mathrm{q}\Big|_{l/3}^{2l/3} + N_0^{(3)} \cdot \mathrm{q}\Big|_{2l/3}^{l} + \int_{l/3}^{2l/3} \frac{dN_1^{(2)}}{dx} Q \cdot dx + \int_{2l/3}^{l} \frac{dN_0^{(3)}}{dx} \cdot Q dx$$

For i = 3

$$\int_{2l/3}^{l/3} \frac{dN_1^{(3)}}{dx} \cdot \left[\frac{dN_0^{(3)}}{dx} \cdot a_2 + \frac{dN_1^{(3)}}{dx} \cdot a_3\right] \cdot dx = N_1^{(3)} \cdot \mathrm{q}\Big|_{2l/3}^{l} + \int_{2l/3}^{l} \frac{dN_1^{(3)}}{dx} Q \cdot dx$$

The expression can be written in matrix form by assembling each matrix element $K^{(e)}$.

The general form to get the element of the K matrix and *f in* in local numbering are:

$$K_{ij}^{\mathrm{e}} = \int_{(1)}^{(2)} \frac{dN_i^e}{dx} \cdot \frac{dN_j^{\mathrm{e}}}{dx} dx \qquad f_i^{\mathrm{e}} = \int_{(1)}^{(2)} N_i^{\mathrm{e}}(x)\, Q\, dx$$

The shape function takes the value 1 at node *i* and the value 0 at the other node.

Knowing the value of:

$$\frac{dN_0^e(x)}{dx} = -\frac{1}{l^e}$$

$$\frac{dN_1^e(x)}{dx} = \frac{1}{l^e}$$

Where l^e is the length of each element.

For each element the following matrixes are obtained:

$$K^0 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$K^1 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$K^2 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$K^3 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

By assembling the all matrixes the global stiffness matrix is:

$$K = 3 \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

The equations of each shape function are:

$$N_0^1(x) = -3x + 1$$

$$N_1^1(x) = 3x$$

$$N_0^2(x) = -3x + 2$$

$$N_1^2(x) = 3x - 1$$

$$N_0^3(x) = -3x + 3$$

$$N_1^3(x) = 3x - 2$$

Taking Q(x) = sin(x), f_i is obtained as:

$$f^0 = \int_{(0)}^{(1/3)} (-3\mathrm{x} + 1) \cdot \sin(\mathrm{x})\ dx$$

$$f^1 = \int_{(0)}^{(1/3)} (3\mathrm{x}) \cdot \sin(\mathrm{x}) dx + \int_{(1/3)}^{(2/3)} (-3\mathrm{x} + 2) \cdot \sin(\mathrm{x}) dx$$

$$f^2 = \int_{(1/3)}^{(2/3)} (3\mathrm{x} - 1) \cdot \sin(\mathrm{x}) dx + \int_{(2/3)}^{(1)} (-3\mathrm{x} + 3) \cdot \sin(\mathrm{x}) dx$$

$$f^3 = \int_{(2/3)}^{(1)} (3\mathrm{x} - 2) \cdot \sin(\mathrm{x})\ dx$$

By integrating by part the above integrals, the system of equation reads:

$$3 \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -q_0 + f^0 \\ f^1 \\ f^2 \\ f^3 + q_l \end{pmatrix}$$

Using the boundary condition we can simplify the matrix considering that the values of u_0 and u_3 as:

$$3 \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 6\sin\left(\frac{1}{3}\right) - 3\sin\left(\frac{2}{3}\right) \\ 6\sin\left(\frac{2}{3}\right) - 3\sin\left(\frac{1}{3}\right) - 3\sin(1) + 3 \cdot \alpha \end{pmatrix}$$

The results by solving the system of equations are:

$$u_1 = 1.0467$$

$$u_2 = 2.0574$$

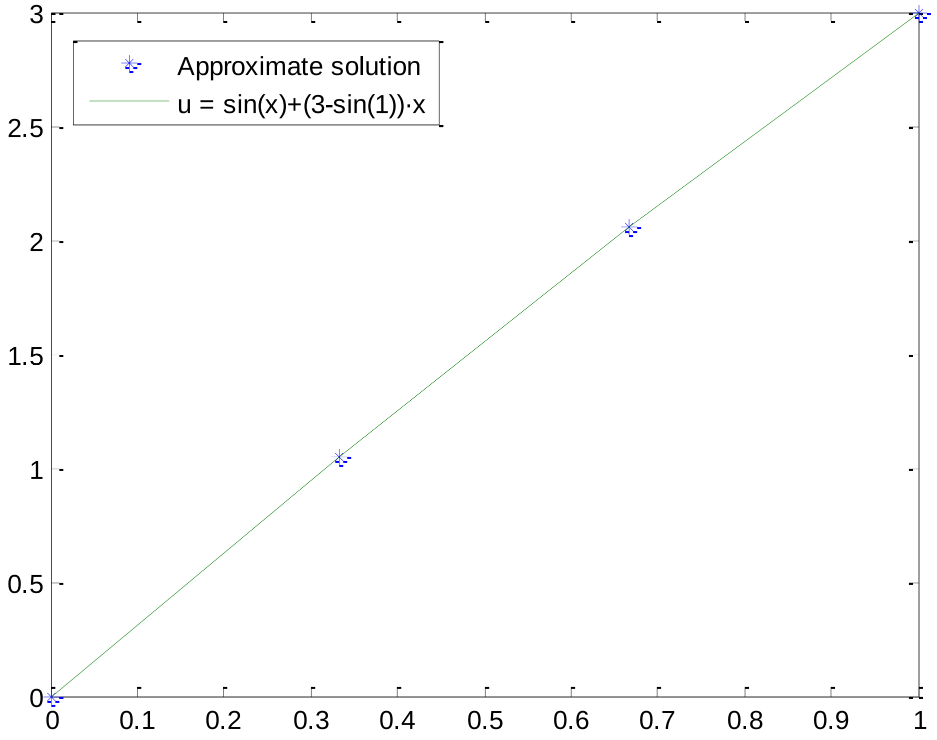
With these values we obtain:

$$q_0 = 3.1585$$

$$q_l = 1.7164$$

Where q_0 and q_l are the reaction fluxes.

The picture shows the comparison between the analitic solution and the approximation solution, where the analytical solution is



The table shows the values of the analytical function and the approximate function at each node.

	x=0	x=1/3	x=2/3	x=1
u(x)	0	1.0467	2.0574	3
u^h(x)	0	1.0467	2.0573	3