Consider the following differential equation:

$$-u'' = f \ in \ ]0,1[$$

Find the weak form of the problem. Describe the FE approximation u^h

$$u^h = \sum_{i=0}^{n} a_i N_i(x)$$

We can express the approximation u^h as the linear combination of function N_i

$$\frac{d^2u}{dx^2} \cong \frac{d^2u^h}{dx^2} = \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{n} a_i N_i(x)\right)$$

The integral form is:

$$-\int_0^1 w\left(\frac{d^2u}{dx^2}\right)dx = \int_0^1 w\, f dx$$

$$\int_0^1 \frac{\partial w}{\partial x}\left(\frac{du}{dx}\right)dx = \int_0^1 w\, f dx \quad \forall w$$

$$\int_0^1 \frac{\partial w}{\partial x}\left(\frac{dN_j}{dx}a_j\right)dx = \int_0^1 w\, f dx \quad \forall w$$

The interpolation function w is going to be $w_i = N_i$ using Galerkin method.

$$\underbrace{\int_0^1 \frac{dN_i}{dx}\left(\frac{dN_j}{dx}a_j\right)}_{Matrix\ K_{ij}\cdot vector\ a_j} dx = \underbrace{\int_0^1 N_i\, f dx + N_i \frac{du}{dx}\bigg|_1 - N_i \frac{du}{dx}\bigg|_0}_{Vector\ f_i}$$

$$\boldsymbol{Ka} = \boldsymbol{f}$$

This is the linear system that we need to solve to obtain the approximation of u^h

$$k_{ij} = \int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx$$

$$f_i = \int_0^1 N_i\, f dx + N_i \frac{du}{dx}\bigg|_1 - N_i \frac{du}{dx}\bigg|_0$$

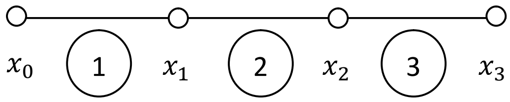
As $N_i|_0 = 0$ and $N_i|_1 = 1$

$$f_i = \int_0^1 N_i\, f dx + \frac{du}{dx}\bigg|_1 = \int_0^1 N_i\, f dx - \bar{q}$$

The finite element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0,1,\dots,n$ and $h = \frac{1}{n}$ we have to compute the approximation u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$

So we get an element divided in three equal parts where:

$$x_0 = 0; \quad x_1 = \frac{1}{3}\ ; \quad x_2 = \frac{2}{3}\ ; \ x_3 = 1$$



$$N_1^{(e)} = \frac{x_2^{(e)} - x}{h^{(e)}};\ N_2^{(e)} = \frac{x - x_1^{(e)}}{h^{(e)}}$$

Where $h^{(e)} = \frac{1}{3}$ for all the elements. And the derivatives are the following:

$$\frac{dN_1}{dx} = -\frac{1}{h^e}; \qquad \frac{dN_2}{dx} = \frac{1}{h^e}$$

$$u^h = N_1 a_1 + N_2 a_2 + N_3 a_3$$

We have to solve the system $\boldsymbol{Ka} = \boldsymbol{f}$ for $n = 3$:

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} - \overline{q_0} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + \overline{q_1} \end{bmatrix}$$

We need to compute all the k_{ij} for the three elements that we have.

$$k_{11}^{(1)} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx} dx \int_0^{1/3} \frac{1}{(1/3)^2} dx = 9 \cdot x|_0^{1/3} = 3 = k_{22}^{(1)}$$

$$k_{12}^{(1)} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx} dx \int_0^{1/3} \frac{-1}{(1/3)^2} dx = -9 \cdot x|_0^{1/3} = -3 = k_{21}^{(1)}$$

$$k_{11}^{(2)} = \int_{1/3}^{2/3} \frac{dN_1}{dx}\frac{dN_1}{dx} dx \int_{1/3}^{2/3} \frac{1}{(1/3)^2} dx = 9 \cdot x|_{1/3}^{2/3} = 3 = k_{22}^{(2)}$$

$$k_{12}^{(2)} = \int_{1/3}^{2/3} \frac{dN_1}{dx}\frac{dN_2}{dx} dx \int_{1/3}^{2/3} \frac{-1}{(1/3)^2} dx = -9 \cdot x|_{1/3}^{2/3} = -3 = k_{21}^{(2)}$$

$$k_{11}^{(3)} = \int_{2/3}^{1} \frac{dN_1}{dx}\frac{dN_1}{dx} dx \int_{2/3}^{1} \frac{1}{(1/3)^2} dx = 9 \cdot x|_{2/3}^{1} = 3 = k_{22}^{(3)}$$

$$k_{12}^{(3)} = \int_{2/3}^{1} \frac{dN_1}{dx}\frac{dN_2}{dx} dx \int_{2/3}^{1} \frac{-1}{(1/3)^2} dx = -9 \cdot x|_{2/3}^{1} = -3 = k_{21}^{(3)}$$

We also need to compute $Vector\ f_i$ for the three elements:

$$f_1^{(1)} = \int_0^{1/3} N_1 \cdot \sin x\, dx = \int_0^{1/3} \frac{x_2^{(1)} - x}{h^{(1)}} \cdot \sin x\, dx$$
$$= -\frac{x_2^{(1)}}{h^{(1)}} \cdot \cos x \Big|_0^{1/3} - \frac{1}{h^{(1)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_0^{1/3} = -1 - 3 \cdot sin(1/3)$$
$$= 0.0184$$

$$f_2^{(1)} = \int_0^{1/3} N_2 \cdot \sin x\, dx = \int_0^{1/3} \frac{x - x_1^{(1)}}{h^{(1)}} \cdot \sin x\, dx$$
$$= \frac{1}{h^{(1)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_0^{1/3} + \frac{x_1^{(1)}}{h^{(1)}} \cdot \cos x \Big|_0^{1/3} = 3 \cdot \sin(1/3) - \cos(1/3)$$
$$= 0.0366$$

$$f_1^{(2)} = \int_{\frac{1}{3}}^{\frac{2}{3}} N_1^{(2)} \cdot \sin x\, dx = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{x_3^{(2)} - x}{h^{(2)}} \cdot \sin x\, dx = -\frac{x_3^{(2)}}{h^{(2)}} \cdot \cos x \Big|_{\frac{1}{3}}^{\frac{2}{3}} - \frac{1}{h^{(2)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_{\frac{1}{3}}^{\frac{2}{3}} =$$
$$\cos\frac{1}{3} - 3 \cdot \left(\sin\frac{2}{3} + \sin\frac{1}{3}\right) = 0.0714$$

$$f_2^{(2)} = \int_{1/3}^{2/3} N_2^{(2)} \cdot \sin x\, dx = \int_{1/3}^{2/3} \frac{x - x_2^{(2)}}{h^{(2)}} \cdot \sin x\, dx$$
$$= \frac{x_2^{(2)}}{h^{(2)}} \cdot \cos x \Big|_{1/3}^{2/3} + \frac{1}{h^{(2)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_{1/3}^{2/3}$$
$$= -\cos\frac{2}{3} + 3 \cdot \left(\sin\frac{2}{3} - \sin\frac{1}{3}\right) = 0.0876$$

$$f_1^{(3)} = \int_{1/3}^{2/3} N_1^{(3)} \cdot \sin x\, dx = \int_{2/3}^{1} \frac{x_4^{(3)} - x}{h^{(3)}} \cdot \sin x\, dx$$
$$= -\frac{x_4^{(3)}}{h^{(3)}} \cdot \cos x \Big|_{2/3}^{1} - \frac{1}{h^{(3)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_{2/3}^{1}$$
$$= \cos\frac{2}{3} - 3 \cdot \left(\sin 1 - \sin\frac{2}{3}\right) = 0.1163$$

$$f_2^{(3)} = \int_{1/3}^{2/3} N_2^{(3)} \cdot \sin x\, dx = \int_{2/3}^{1} \frac{x - x_3^{(3)}}{h^{(3)}} \cdot \sin x\, dx$$
$$= \frac{x_3^{(3)}}{h^{(3)}} \cdot \cos x \Big|_{2/3}^{1} + \frac{1}{h^{(3)}} \cdot (\sin x - \mathrm{x} \cdot \cos x)\Big|_{2/3}^{1}$$
$$= -\cos 1 + 3 \cdot \left(\sin 1 - \sin\frac{2}{3}\right) = 0.129$$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0.0184 - \overline{q_0} \\ 0.108 \\ 0.2042 \\ 0.129 + \overline{q_1} \end{bmatrix}$$

I know for the boundary conditions that $a_1 = u(0) = 0$ and $a_4 = u(1) = \alpha = 3$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ a_2 \\ a_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 - \overline{q_0} \\ 0.108 \\ 0.2042 \\ 0.129 + \overline{q_1} \end{bmatrix}$$

$$a_2 = 1.0467$$

$$a_3 = 2.057$$

To compare it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$ we have to compute $u\left(\frac{1}{3}\right)$ *and* $u\left(\frac{2}{3}\right)$

$$u\left(\frac{1}{3}\right) = \sin\frac{1}{3} + \frac{(3 - \sin 1)1}{3} = 1.0467$$

$$u\left(\frac{2}{3}\right) = \sin\frac{2}{3} + \frac{(3 - \sin 1)2}{3} = 2.0573$$

As we can see the approximate solution is equal to the exact solution.