Homework 1

Basics of FE

Finite Elements Method

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1. Find the weak form of the problem. Describe the FE approximation $\mathbf{u}^h$.

Considering the following differential equation:

$$-u'' = f(x)$$

The integral form of the problem is:

$$-\int_0^1 v\frac{\partial^2 u}{\partial x^2}dx = \int_0^1 vf(x)dx$$

Integrate by part:

$$-\int_0^1 v\frac{\partial^2 u}{\partial x^2}dx = -\left. v\frac{\partial u}{\partial x}\right|_0^1 + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx$$

The weak form of the problem reads:

$$\int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx = \left. v\frac{\partial u}{\partial x}\right|_0^1 + \int_0^1 vf(x)dx$$

$$u^h(x) = \sum_{i=0}^{n} N_i(x)a_i$$

$$v_i(x) = N_i(x)$$

2. Describe the linear system of equations to be solved

Discretization of the weak form

$$\int_0^1 \frac{\partial N_i}{\partial x}\sum_{j=0}^{n}\frac{\partial N_j}{\partial x}a_j dx = N_i q|_0^1 + \int_0^1 N_i f(x)dx, i = 0, 1, 2, ...n$$

$$\begin{bmatrix} K_{00} & K_{01} & \cdots & K_{0n} \\ K_{10} & K_{11} & \cdots & K_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n0} & K_{n1} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{Bmatrix}$$

Ka=f

$$k_{ij} = \int_0^1 \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dx,$$
$$i = 0, 1, 2, ...n,$$
$$j = 0, 1, 2, ...n$$
$$f_i = N_i q|_0^1 + \int_0^1 N_i f(x)dx$$

3. Compute the FE approximation u^h for n=3, $f(x) = sin(x)$ and $\alpha = 3$. Compare it with the exact solution.

We will now solve the problem using a discretization of the bar into three linear elements. The approximation of u^h over the whole bar can be written as

$$u^h(x) = N_0(x)a_0 + N_1(x)a_1 + N_2(x)a_2 + N_3(x)a_3$$

where N_i are called global shape function. Now we are going to obtain the local shapes function.

$$\begin{matrix} 0 \le x \le 1/3 \\ 1/3 \le x \le 2/3 \\ 2/3 \le x \le 1 \end{matrix} \begin{cases} N_0 = N_1^{(1)} \\ N_0 = 0 \\ N_0 = 0 \end{cases}$$

$$\begin{matrix} 0 \le x \le 1/3 \\ 1/3 \le x \le 2/3 \\ 2/3 \le x \le 1 \end{matrix} \begin{cases} N_1 = N_2^{(1)} \\ N_1 = N_1^{(2)} \\ N_1 = 0 \end{cases}$$

$$\begin{matrix} 0 \le x \le 1/3 \\ 1/3 \le x \le 2/3 \\ 2/3 \le x \le 1 \end{matrix} \begin{cases} N_2 = 0 \\ N_2 = N_2^{(2)} \\ N_2 = N_1^{(3)} \end{cases}$$

$$\begin{matrix} 0 \le x \le 1/3 \\ 1/3 \le x \le 2/3 \\ 2/3 \le x \le 1 \end{matrix} \begin{cases} N_3 = 0 \\ N_3 = 0 \\ N_3 = N_2^{(3)} \end{cases}$$

The discretized weak form is written now:

$$\int_0^1 \frac{dN_i}{dx}[\frac{dN_0}{dx}a_0 + \frac{dN_1}{dx}a_1 + \frac{dN_2}{dx}a_2 + \frac{dN_3}{dx}a_3]dx = \int_0^1 N_i f(x)dx + N_i q|_0^1, i = 0, 1, 2, 3.$$

The global solution system written in local shape function:

$$\int_0^{1/3} \frac{dN_1^{(1)}}{dx}[\frac{dN_1^{(1)}}{dx}a_0 + \frac{dN_2^{(1)}}{dx}a_1]dx = \int_0^{1/3} N_1^{(1)} f(x)dx + q_1$$

$$\int_0^{1/3} \frac{dN_2^{(1)}}{dx}[\frac{dN_1^{(1)}}{dx}a_0 + \frac{dN_2^{(1)}}{dx}a_1]dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx}[\frac{dN_1^{(2)}}{dx}a_1 + \frac{dN_2^{(2)}}{dx}a_2]dx =$$

$$= \int_0^{1/3} N_2^{(1)} f(x)dx + \int_{1/3}^{2/3} N_1^{(2)} f(x)dx$$

$$\int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx}[\frac{dN_1^{(2)}}{dx}a_1 + \frac{dN_2^{(2)}}{dx}a_2] + \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx}[\frac{dN_1^{(3)}}{dx}a_2 + \frac{dN_2^{(3)}}{dx}a_3]dx =$$

$$= \int_{1/3}^{2/3} N_2^{(2)} f(x)dx + \int_{2/3}^{1} N_1^{(3)} f(x)dx$$

$$\int_{2/3}^{1} \frac{dN_2^{(3)}}{dx}[\frac{dN_1^{(3)}}{dx}a_2 + \frac{dN_2^{(3)}}{dx}a_3]dx = \int_{2/3}^{1} N_2^{(3)} f(x)dx + q_4$$

The above expressions can be written in matrix form as:

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & \\ & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ & & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{Bmatrix}$$

$$\begin{array}{l} N_1^{(1)} = \frac{x_1 - x}{l^{(1)}};\ N_1^{(2)} = \frac{x_2 - x}{l^{(2)}};\ N_1^{(3)} = \frac{x_3 - x}{l^{(3)}} \\ N_2^{(1)} = \frac{x - x_0}{l^{(1)}};\ N_2^{(2)} = \frac{x - x_1}{l^{(2)}};\ N_2^{(3)} = \frac{x - x_2}{l^{(3)}} \\ \frac{dN_1^{(1)}}{dx} = \frac{-1}{l^{(1)}};\ \frac{dN_1^{(2)}}{dx} = \frac{-1}{l^{(2)}};\ \frac{dN_1^{(3)}}{dx} = \frac{-1}{l^{(3)}} \\ \frac{dN_2^{(1)}}{dx} = \frac{1}{l^{(1)}};\ \frac{dN_2^{(2)}}{dx} = \frac{1}{l^{(2)}};\ \frac{dN_2^{(3)}}{dx} = \frac{1}{l^{(3)}} \end{array}$$

Substituting the above expressions into $k_{ij}^{(e)}$, we obtain:

$$k_{11}^{(1)} = \int_0^{1/3} \frac{dN_1^{(1)}}{dx}\frac{dN_1^{(1)}}{dx} = \frac{1}{3l^2} = 3 = k_{11}^{(2)} = k_{11}^{(3)}$$

$$k_{22}^{(1)} = \int_0^{1/3} \frac{dN_2^{(1)}}{dx}\frac{dN_2^{(1)}}{dx} = \frac{1}{3l^2} = 3 = k_{22}^{(2)} = k_{22}^{(3)}$$

$$k_{12}^{(1)} = \int_0^{1/3} \frac{dN_1^{(1)}}{dx}\frac{dN_2^{(1)}}{dx} = -\frac{1}{3l^2} = -3 = k_{12}^{(2)} = k_{12}^{(3)}$$

Computing $f_i^{(e)}$, we obtain:

$$f_1^{(1)} = \int_0^{1/3} N_1^{(1)} sin(x)dx + q_1 = \int_0^{1/3} \frac{x_1 - x}{l} sin(x)dx + q_1 = 0.018 + q_1$$

$$f_2^{(1)} = \int_0^{1/3} N_2^{(1)} sin(x)dx = \int_0^{1/3} \frac{x - x_0}{l} sin(x)dx = 0.037$$

$$f_1^{(2)} = \int_{1/3}^{2/3} N_1^{(2)} sin(x)dx = \int_{1/3}^{2/3} \frac{x_2 - x}{l} sin(x)dx = 0.071$$

$$f_2^{(2)} = \int_{1/3}^{2/3} N_2^{(2)} sin(x)dx = \int_{1/3}^{2/3} \frac{x - x_1}{l} sin(x)dx = 0.088$$

$$f_1^{(3)} = \int_{2/3}^{1} N_1^{(3)} sin(x)dx = \int_{2/3}^{1} \frac{x_3 - x}{l} sin(x)dx = 0.117$$

$$f_2^{(3)} = \int_{2/3}^{1} N_2^{(3)} sin(x)dx + q_4 = \int_{2/3}^{1} \frac{x - x_2}{l} sin(x)dx + q_4 = 0.129 + q_4$$

The global matrix equation is therefore written as:

$$\begin{bmatrix} 3 & -3 & & \\ -3 & 6 & -3 & \\ & -3 & 6 & -3 \\ & & -3 & 3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0.018 + q_1 \\ 0.108 \\ 0.204 \\ 0.129 + q_4 \end{Bmatrix}$$

The solution of the above system gives ($a_0 = 0; a_3 = \alpha = 3$):

$$q_1 = -1.007;\ q_4 = -0.820;\ a_1 = 1.059 : a_2 = 2.051$$

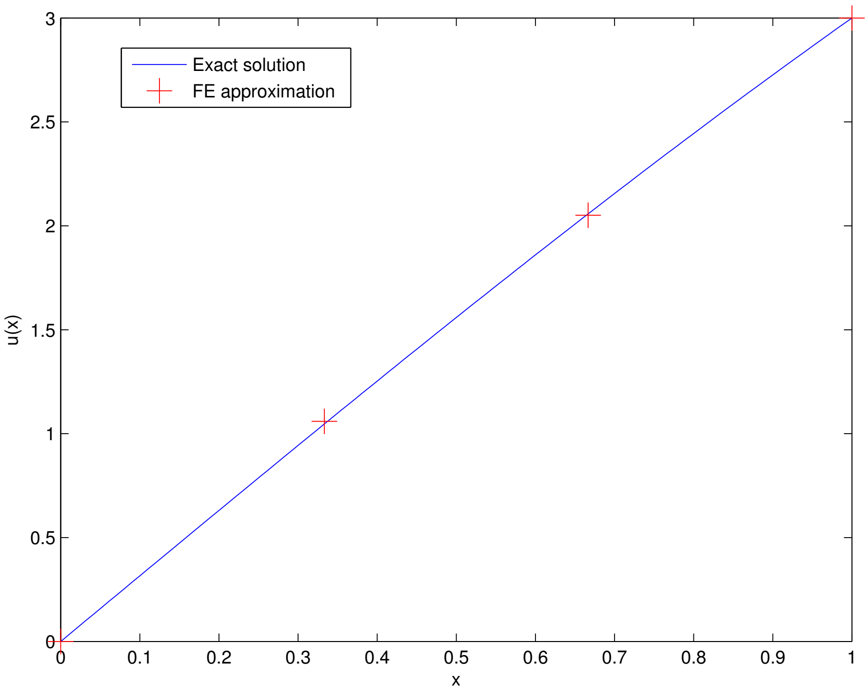

Figure 1: Comparison of exact solution and FE approximation.