FEM Homework 2 ARTEMII SATTAROV

The strong form of the problem:

$$\mathrm{div}\,\sigma + b = 0$$

$$u = \bar{u} \text{ on } \Omega$$

Derivation of a weak form of the problem:
In our case the only one external force is equally distributed body force b per area.

Applying principle of virtual work to the element we obtain equilibrating nodal forces:

$$\iint_{A^e} \delta\varepsilon^T \sigma t\, dA - \iint_{A^e} \delta u^T b t\, dA = [\delta a^e]^T q^e \quad (1)$$

where $[\delta a^e]^T q^e$ - virtual work by equilibrating nodal forces.

Interpolation of the virtual displacements in terms of nodal values:

$$\delta u^T = [\delta a^e]^T N^T \;;\quad \delta\varepsilon^T = [\delta a^e]^T B^T$$

Substituting these values into eq. (1) gives:

$$[\delta a^e]^T \left(\iint_{A^e} B^T \sigma t\, dA - \iint_{A^e} N^T b t\, dA \right) = [\delta a^e]^T q^e$$

Canceling virtual displacements and substituting stress without consideration of initial strains and stresses:

$$\iint_{A^e} B^T D B t\, dA \cdot a^e - \iint_{A^e} N^T b t\, dA = q^e$$

Expression for the global equilibrium:

$$K \cdot a = f$$

where K - global stiffness matrix; f - equilibrating nodal force

Boundary conditions:

Dirichlet B.C with zero displacements for nodes 1,2,3 and with zero horizontal displacement for nodes 6,5:

$V_1 = 0$

$U_1 = 0$

$V_2 = 0$

$U_2 = 0$

$V_3 = 0$

$U_3 = 0$

$U_5 = 0$

$U_6 = 0$

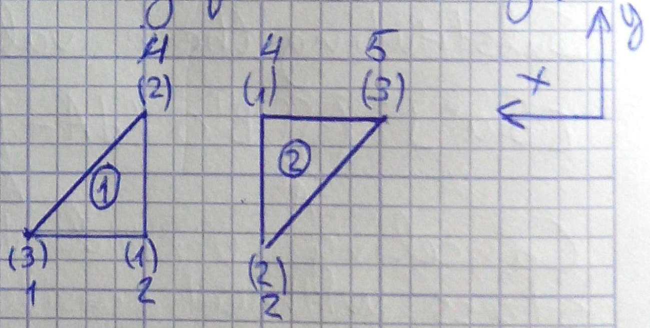
Dirichlet B.C with prescribed displacement for node 6:

$V_6 = -\delta = -0{,}01\ m$

2. Mesh description. Let's introduce local numbering for the elements, starting from the right angle

	(1)	(2)	(3)
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

– connectivity matrix



3. To build linear system of equations, we found local stiffness matrices and assembled into a global one:

$$\begin{array}{c}\\ 2\\ 4\\ 1\end{array}\begin{pmatrix} \overset{2}{K_{11}^{(1)}} & \overset{4}{K_{12}^{(1)}} & \overset{1}{K_{13}^{(1)}} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{pmatrix} \quad \begin{array}{c}\\ 3\\ 5\\ 2\end{array}\begin{pmatrix} \overset{3}{K_{11}^{(3)}} & \overset{5}{K_{12}^{(3)}} & \overset{2}{K_{13}^{(3)}} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} \end{pmatrix} \quad \begin{array}{c}\\ 5\\ 6\\ 4\end{array}\begin{pmatrix} \overset{5}{K_{11}^{(4)}} & \overset{6}{K_{12}^{(4)}} & \overset{4}{K_{13}^{(4)}} \\ K_{21}^{(4)} & K_{22}^{(4)} & K_{23}^{(4)} \\ K_{31}^{(4)} & K_{32}^{(4)} & K_{33}^{(4)} \end{pmatrix} \quad \begin{array}{c}\\ 4\\ 2\\ 5\end{array}\begin{pmatrix} \overset{4}{K_{11}^{(2)}} & \overset{2}{K_{12}^{(2)}} & \overset{5}{K_{13}^{(2)}} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{pmatrix}$$

1st element 3rd element 4th element 2nd element

A typical stiffness submatrix element:

$$K_{ij}^{(e)} = \iint_{A^e} B_i^T D B_j \, t \, dA = \iint_{A^e} \frac{1}{2A^e} \begin{bmatrix} b_j & 0 & c_j \\ 0 & c_j & b_j \end{bmatrix} \begin{vmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{vmatrix} \frac{1}{2A^e} \cdot \begin{vmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{vmatrix} t \, dA$$

$$K_{ij}^{e} = \left(\frac{t}{4A}\right)^e \begin{vmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i c_j d_{21} + b_i b_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{vmatrix}$$

Linear system of equations $K \cdot a = f$, which is as follows:

$$\begin{vmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{11}^{(1)}+K_{33}^{(3)}+K_{22}^{(2)} & K_{31}^{(3)} & K_{12}^{(1)}+K_{21}^{(2)} & K_{32}^{(3)}+K_{23}^{(2)} & 0 \\ 0 & K_{31}^{(3)} & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ K_{32}^{(1)} & K_{12}^{(1)}+K_{21}^{(2)} & 0 & K_{22}^{(1)}+K_{33}^{(4)}+K_{11}^{(2)} & K_{31}^{(4)}+K_{13}^{(2)} & K_{32}^{(4)} \\ 0 & K_{32}^{(3)}+K_{23}^{(2)} & K_{12}^{(3)} & K_{31}^{(4)}+K_{13}^{(2)} & K_{22}^{(3)}+K_{11}^{(4)}+K_{33}^{(2)} & K_{12}^{(4)} \\ 0 & 0 & 0 & K_{32}^{(4)} & K_{12}^{(4)} & K_{22}^{(4)} \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ V_4 \\ 0 \\ V_5 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} r_1 + f_3^{(1)} \\ r_2 + f_1^{(1)} + f_2^{(2)} + f_3^{(3)} \\ r_3 + f_1^{(3)} \\ f_2^{(4)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} + r_5 \\ f_2^{(4)} + r_6 \end{vmatrix}$$

Stiffness matrix consists of 12 rows and 12 columns

After application of boundary conditions the system to be solved has 3 degree of freedom.

4. To calculate FE approximations we firstly compute local stiffness matrices of the elements

$$K_{11}^{(e)} = \frac{1}{4A} \cdot 10^{10} \begin{vmatrix} 3,28125 & 1,40625 \\ 1,40625 & 3,28125 \end{vmatrix}$$

$b_1 = 1,5$ $c_1 = 1,5$

$b_2 = 0$ $c_2 = -1,5$

$b_3 = -1,5$ $c_3 = 0$

$d_{11} = d_{22}$

Using plane stress model:

$$d_{11} = d_{22} = \frac{E}{1-\vartheta^2} = 1,04166(6) \cdot 10^{10}$$

$$d_{12} = d_{21} = \vartheta d_{11} = 0,2083(3) \cdot 10^{10}$$

$$d_{33} = \frac{E}{2(1+\vartheta)} = 0,416(6) \cdot 10^{10}$$

$$K_{12}^{(e)} = \frac{1}{4A} \cdot \not{10^{10}} \begin{vmatrix} -0,9375 \cdot 10^{10} & -0,46875 \cdot 10^{10} \\ -0,9375 \cdot 10^{10} & -2,34375 \cdot 10^{10} \end{vmatrix}$$

$$K_{21}^{(e)} = \frac{1}{4A} \cdot \not{10^{10}} \begin{vmatrix} -0,9375 \cdot 10^{10} & -0,9375 \cdot 10^{10} \\ -0,46875 \cdot 10^{10} & -2,34375 \cdot 10^{10} \end{vmatrix}$$

$$K_{22}^{(e)} = \frac{1}{4A} \cdot \not{10^{10}} \begin{vmatrix} 0,9375 \cdot 10^{10} & 0 \\ 0 & 2,34375 \cdot 10^{10} \end{vmatrix}$$

$$K_{23}^{(e)} = \frac{1}{4A} \cdot \not{10^{10}} \begin{vmatrix} 0 & 0,9375 \cdot 10^{10} \\ 0,46875 \cdot 10^{10} & 0 \end{vmatrix}$$

$$K_{32}^{(e)} = \frac{1}{4A} \begin{vmatrix} 0 & 0,46875 \cdot 10^{10} \\ 0,9375 \cdot 10^{10} & 0 \end{vmatrix}$$

$$K_{33}^{(e)} = \frac{1}{4A} \begin{vmatrix} 2,34375 \cdot 10^{10} & 0 \\ 0 & 0,9375 \cdot 10^{10} \end{vmatrix}$$

$$K_{31}^{(e)} = \frac{1}{4A} \begin{vmatrix} -2,34375 \cdot 10^{10} & -0,46875 \cdot 10^{10} \\ -0,9375 \cdot 10^{10} & -0,9375 \cdot 10^{10} \end{vmatrix}$$

$$K_{13}^{(e)} = \frac{1}{4A} \begin{vmatrix} -2,34375 \cdot 10^{10} & -0,9375 \cdot 10^{10} \\ -0,46875 \cdot 10^{10} & -0,9375 \cdot 10^{10} \end{vmatrix}$$

$$A = \frac{1,5 \cdot 1,5}{2} = 1,125$$

The body force element $f_i^{(e)}$ is equal for each element. As body force equally spaced between 3 nodes of the element we obtain:

$$f_i^{(e)} = \frac{(A^t t)^e}{3} \begin{vmatrix} b_x \\ b_y \end{vmatrix} = \begin{vmatrix} 0 \\ -375 \end{vmatrix} \quad (b_y = -3g)$$

Eliminating rows and columns from stiffness matrix and moving $K_{32}^{(4)} \cdot \delta$; $K_{12}^{(4)} \cdot \delta$; $K_{22}^{(4)} \cdot \delta$ components to the right hand side, we obtained:

$$\begin{vmatrix} K_{22}^{(1)} + K_{33}^{(4)} + K_{11}^{(2)} & K_{31}^{(4)} + K_{13}^{(2)} \\ K_{31}^{(4)} + K_{13}^{(2)} & K_{22}^{(3)} + K_{11}^{(4)} + K_{33}^{(2)} \end{vmatrix} \begin{vmatrix} U4 \\ V4 \\ V5 \end{vmatrix} = \begin{vmatrix} 0 - \delta \cdot \frac{0,46875 \cdot 10^{10}}{4 \cdot 1,125} \\ -1125 - 0 \\ -1125 - \delta \cdot \frac{-2,34375 \cdot 10^{10}}{4 \cdot 1,125} \end{vmatrix}$$

Substituting K_{ij} values and solving linear system of 3 equations by inverse matrix method we obtain values of unknowns:

$U4 = -0,000128205$ m.

$V4 = -0,001132587$ m.

$V5 = -0,003867629$ m.