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The Finite Element Method
Master in Numerical Methods in Engineering

Consider the following differential equation:

$$(1)\ \ -u'' = f\ in\ ]0,1[$$

With boundary conditions, $u(0) = 0\ and\ u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih\ for\ i = 0,1,\dots,n\ and\ h = \frac{1}{n}$.

a) Find the weak form of the problem. Describe the FE approximation u^h.
b) Describe the linear system to be solved

In order to "relax" the equation of the problem aforementioned, we multiply by a weight function and we integrate the resultant expression over the whole domain between 0 and 1.

$$(2)\ \ \int_0^1 W(x)\cdot\left(\frac{d^2u}{dx^2}+f\right)dx = 0$$

Computationally talking is not possible to obtain a continuous function as a solution of a problem; therefore we look for some function which states for the solution of the problem but being slightly different. The approximated solution is called u^h and it is a piecewise polynomial function constituted as follows:

$$(3)\ \ u \approx u^h = \sum_{j=1}^{n} N_j a_j$$

Where j is the number of elements as many as n and, $N_j(x)$ is the shape function and has value 1 in j=1 and is equal to 0 in the remaining nodes; the coefficients a_j are the unknowns of the problem. If we plug the approximate solution u^h into (2) arises the weight residual expression of the problem. The next step is, by using the expression (3) in the weighted residual expression and considering the equality valid for each element we read:

$$(4)\ \ \int_0^1 W_i(x)\cdot\frac{d^2}{dx^2}\left(\sum_{j=0}^{n} N_j a_j\right)dx\ +\int_0^1 W_i(x)\cdot f\ dx = 0$$

In order to obtain the weak form of the problem we should integrate by parts the first term in the equation (2) by considering that, u = W and $\frac{d^2u}{dx^2}$ = dv. Hence, using the resultant expression and substituting we get:

$$(5)\ \ \int_0^1 \frac{du^h}{dx}\cdot\frac{dW_i(x)}{dx}dx = W_i(x)\cdot\frac{du^h}{dx}\bigg|_0^1 + \int_0^1 W_i(x)\,f\ dx$$

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Now we select the Garlekin method to solve the problem which states that $W_i = N_i$ and we put the solution u^h in (5) in terms of the shape functions to obtain:

$$(6)\ \int_0^1 \frac{dN_i}{dx} \cdot \frac{dN_j(x)}{dx} a_j dx = N_i(x) \cdot \frac{du^h}{dx}\bigg|_0^1 + \int_0^1 N_i(x)\, f_i\, dx$$

It should be outlined that the following term is equal to the difference of flux between the two limit nodes of the domain.

$$(7)\ N_i(x) \cdot \frac{du^h}{dx}\bigg|_0^1 = q_0 - q$$

From this point we must solve a system of n*n which has the same number of equations as unknowns and has the form $Ka = f$ where K is the stiffness matrix, and the vector of unknowns and f is a source/sink term.

c) Compute the approximation u^h for n=3, f(x)=sin(x) and α=3. Compare it with the exact solution $u(x) = \sin(x) + (3 - sin1)x$.

From the expression (6) we know that:

$$(8)\ K_{ij}^{\mathrm{e}} = \int_1^2 \frac{dN_i^e}{dx} \cdot \frac{dN_j^{\mathrm{e}}}{dx} dx \qquad (9)\qquad f_i^{\mathrm{e}} = \int_1^2 N_i^{\mathrm{e}}(x)\, f\, dx$$

The nodal coordinates are as follows:

$$x_0 = 0; x_1 = \frac{1}{3}; x_2 = \frac{2}{3};\ x_3 = 1$$

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By taking into consideration that:

$$\frac{dN_0^e(x)}{dx} = -\frac{1}{l^e} \qquad \frac{dN_1^e(x)}{dx} = \frac{1}{l^e}$$

And that $l^e = 1/3$

We get the three local matrix:

$$K^0 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \quad K^1 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \quad K^2 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

Then the three matrix should be assembled into the global stiffness matrix which is the 4*4 system to be solved:

$$\begin{pmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} q_0 + \int_0^{1/3} (-3\mathrm{x}+1) \cdot \sin(\mathrm{x}) \ dx \\ \int_0^{1/3} (3\mathrm{x}) \cdot \sin(\mathrm{x}) dx + \int_{1/3}^{2/3} (-3\mathrm{x}+2) \cdot \sin(\mathrm{x}) dx \\ \int_{1/3}^{2/3} (3\mathrm{x}-1) \cdot \sin(\mathrm{x}) dx + \int_{2/3}^{1} (-3\mathrm{x}+3) \cdot \sin(\mathrm{x}) dx \\ \int_{2/3}^{1} (3\mathrm{x}-2) \cdot \sin(\mathrm{x}) \ dx \ - q \end{pmatrix}$$

In this system u_0 and u_3 are data of the problem equal to the Dirichlet boundary conditions.

The solution of this problem is obtained through the use of Matlab and the solution is:

$$u_1 = 1.0467; \ u_2 = 2.0574; \ q_0 = -\,3.1585; q = \ -1.716$$