Finite Elements - Homework 1

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Given:

$$\frac{\partial^2 u(x)}{\partial x^2} + f(x) = 0$$

- The domain of x is (0,1)
- Boundary condition are $u(0) = 0$ and $u(1) = \alpha$
- The domain consists of n nodes
- There are n-1 elements (2 noded elements) of equal lengths
- The elements are linear (variation of $u(x)$ within the element is linear)

Solution:

To obtain the weak form, we define $w(x)$ such that for any $w(x)$, satisfying the following equation is the necessary and sufficient condition to satisfy the given differential equation.

$$\int_l w \frac{\partial^2 u(x)}{\partial x^2} dx + \int_l w f(x) dx = 0$$

Integrating by parts for the first term,

$$\left[w \frac{\partial u}{\partial x} \right]_{x=0}^{x=1} - \int_l \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dx + \int_l w f dx = 0$$

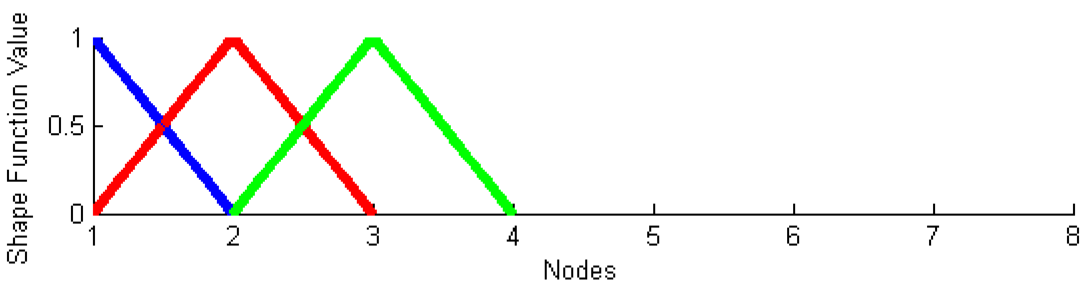
Multiplying by -1 and shifting sides,

$$\int_l \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dx = \int_l w f dx + \left[w \frac{\partial u}{\partial x} \right]_{x=0}^{x=1}$$

This represents the weak form of the given differential function.
Now, u can be approximated by u^h such that,

$$u^h = N_1(x)u_1 + N_2(x)u_2 + ... + N_n(x)u_n = \sum_{j=1}^{n} N_j(x)u_j$$

Where, $u_1, u_2, ..., u_n$ are the values of u(x) at nodes 1,2,...,n respectively. On the other hand, $N_1, N_2, ..., N_n$ are global shape functions defined as follows,



$N_1(x)$ has value 1 at node 1 and it is 0 at all other nodes (shown in blue color)
$N_2(x)$ has value 1 at node 2 and it is 0 at all other nodes (shown in red color)
$N_3(x)$ has value 1 at node 3 and it is 0 at all other nodes (shown in green color) and so on...

Substituting u^h in the weak form,

$$\int_l \frac{\partial w}{\partial x} \frac{\partial \sum_{j=1}^{n} N_j(x) u_j}{\partial x} dx = \int_l w f dx + \left[w \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1}$$

$$\int_l \frac{\partial w}{\partial x} \sum_{j=1}^{n} \frac{\partial N_j}{\partial x} u_j dx = \int_l w f dx + \left[w \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1}$$

In out system, we have n unknowns, u of n-2 nodes and 2 fluxes at both the ends of the domain. Thus we need n equations. Based on the above derivation of weak form, this equation is valid for every value of $w(x)$. So to obtain n equations, we take n functions $w_1, w_2, ..., w_n$ all of which should satisfy the above equation. So we get system of equations which can be written in a compact form like this,

$$\int_l \frac{\partial w_i}{\partial x} \sum_{j=1}^{n} \frac{\partial N_j}{\partial x} u_j dx = \int_l w_i f dx + \left[w_i \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1} \qquad i = 1, 2, 3, ..., n$$

Choosing Galerkin's Method, we take

$$w_i = N_i$$

Substituting in the equation,

$$\int_l \frac{\partial N_i}{\partial x} \sum_{j=1}^{n} \frac{\partial N_j}{\partial x} u_j dx = \int_l N_i f dx + \left[N_i \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1} \qquad i = 1, 2, 3, ..., n$$

The last term in the equation,

$$\left[N_i \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1}$$

for $i = 1$,

$$\left[N_1 \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1} = - \left[\frac{\partial u^h}{\partial x} \right]_{x=0} = -Q_1$$

Since N_1 at $x = 1$ is 0 and N_1 at $x = 0$ is 1
for $i = 2$,

$$\left[N_2 \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1} = 0$$

Since N_2 at $x = 1$ is 0 and N_2 at $x = 0$ is 0
Similarly, all the N_i $i \neq 1$ and $i \neq n$, the above term will be zero.
for $i = n$,

$$\left[N_n \frac{\partial u^h}{\partial x} \right]_{x=0}^{x=1} = \left[\frac{\partial u^h}{\partial x} \right]_{x=1} = Q_n$$

Since N_n at $x = 1$ is 1 and N_n at $x = 0$ is 0
Therefore, the above equation can be written in matrix formulation.

$$\begin{bmatrix} K_{11} & K_{12} & ... & K_{1n} \\ K_{21} & K_{22} & ... & K_{2n} \\ ... & ... & ... & ... \\ K_{n1} & K_{n2} & ... & K_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} = \begin{bmatrix} F_1 - Q_1 \\ F_2 \\ ... \\ F_n + Q_n \end{bmatrix}$$

Where,

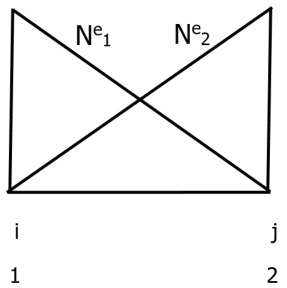
$$K_{ij} = \int_l \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}dx \qquad\qquad F_i = \int_l N_i f dx$$

Now, we can convert the Global Shape Functions to Local Shape Functions. For any general element in domain, 1 and 2 are the local nodes.
N_1^e and N_2^e are the local shape functions defined as,

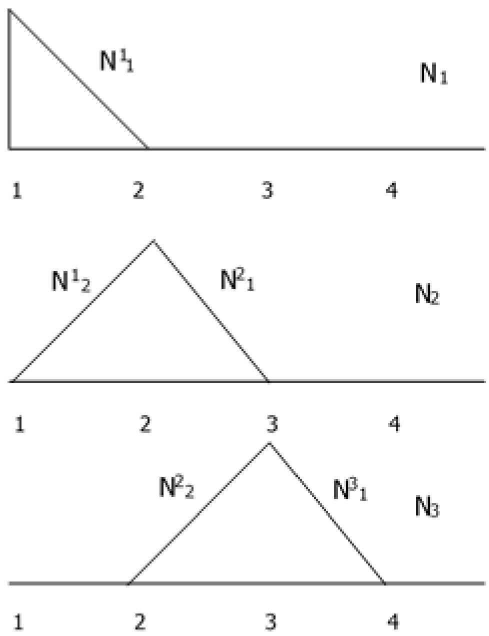
$$N_1^e = \frac{x_2^e - x}{l^e}$$

$$N_2^e = \frac{x - x_1^e}{l^e}$$

n = number of nodes, l^e = the length of the element



Comparing the shapes of global and local shape functions, we get (Note: $h = 1/n$)



Global Functions	$0 \le x \le h$	$h \le x \le 2h$	$2h \le x \le 3h$	...	$1-h \le x \le 1$
N_1	N_1^1	0	0	...	0
N_2	N_2^1	N_1^2	0	...	0
N_3	0	N_2^2	N_1^3	...	0
...	...	...	...	...	...
N_n	0	0	0	...	N_2^{n-1}

Let's calculate different K_{ij}

$$K_{11} = \int_l \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}dx = \int_0^h \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}dx + \int_h^{2h} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}dx$$

$$K_{11} = \int_0^h \frac{\partial N_1^1}{\partial x}\frac{\partial N_1^1}{\partial x}dx + 0 + ... + 0 = K_{11}^1$$

$$K_{12} = \int_l \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x}dx = \int_0^h \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x}dx + \int_h^{2h} \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x}dx$$

$$K_{12} = \int_0^h \frac{\partial N_1^1}{\partial x}\frac{\partial N_2^1}{\partial x}dx + 0 + ... + 0 = K_{12}^1$$

$$K_{13} = \int_l \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x}dx = \int_0^h \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x}dx + \int_h^{2h} \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x}dx = 0$$

Because, N_3 in domain (0,h) is 0 and N_1 in domain (h,1) is 0. Similarly, K_{14}, K_{15},...,K_{1n} are all zero.

$$K_{21} = \int_l \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x}dx = \int_0^h \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x}dx + \int_h^{2h} \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x}dx$$

$$K_{21} = \int_0^h \frac{\partial N_2^1}{\partial x}\frac{\partial N_1^1}{\partial x}dx + 0 + ... + 0 = K_{21}^1$$

$$K_{22} = \int_l \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x}dx = \int_0^h \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x}dx + \int_h^{2h} \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x}dx$$

$$K_{22} = \int_0^h \frac{\partial N_2^1}{\partial x}\frac{\partial N_2^1}{\partial x}dx + \int_h^{2h} \frac{\partial N_1^2}{\partial x}\frac{\partial N_1^2}{\partial x}dx = K_{22}^1 + K_{11}^2$$

Because $N_2 = 0$ in domain (2h,1)

$$K_{23} = \int_l \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x}dx = \int_0^h \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x}dx + \int_h^{2h} \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x}dx + ... + \int_{1-h}^{1} \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x}dx = \int_h^{2h} \frac{\partial N_1^2}{\partial x}\frac{\partial N_2^2}{\partial x}dx = K_{12}^2$$

Because, N_3 in domain (0,h) is 0 and N_2 in domain (2h,1) is 0. Similarly, K_{24}, K_{25},...,K_{2n} are all zero. This is the beauty of the shape functions, the integration over the whole range comes down to the range where both the shape functions are non-zero. Using similar procedure other K_{ij} can be found.

Let's calculate different F_i

Again, due to shape functions property, we don't have to integrate over the whole range, just do it on the range where the shape function is non-zero.

$$F_1 = \int_0^1 N_1 f dx = \int_0^h N_1^1 f dx = F_1^1$$

$$F_2 = \int_0^1 N_2 f dx = \int_0^h N_2^1 f dx + \int_h^{2h} N_1^2 f dx = F_2^1 + F_1^2$$

$$F_3 = \int_0^1 N_3 f dx = \int_h^{2h} N_2^2 f dx + \int_{2h}^{3h} N_1^3 f dx = F_2^2 + F_3^1$$

and so on. Thus the global system of equations now can be written in the local form.

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 & ... & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 & ... & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & K_{22}^{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_n \end{bmatrix} = \begin{bmatrix} F_1^1 - Q_1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ ... \\ F_2^{n-1} + Q_n \end{bmatrix}$$

In the given problem, nodes = 4 (3 elements). So the above matrix can be written as

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1^1 - Q_1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + Q_4 \end{bmatrix}$$

Comment: In the above equation, the only unknowns are $u2$, $u3$, Q_1 and Q_4. Thus, the total unknowns are 4 and we have 4 equations. Let's calculate all the known terms in the matrix.

$$\frac{\partial N_1^e}{\partial x} = -\frac{1}{l^e} = -3$$

$$\frac{\partial N_2^e}{\partial x} = \frac{1}{l^e} = 3$$

Calculating different K_{ij}^e

Element 1

$$K_{11}^1 = \int_0^{1/3} \frac{\partial N_1^1}{\partial x}\frac{\partial N_1^1}{\partial x} dx = 3 \qquad K_{12}^1 = \int_0^{1/3} \frac{\partial N_1^1}{\partial x}\frac{\partial N_2^1}{\partial x} dx = -3$$

$$K_{21}^1 = \int_0^{1/3} \frac{\partial N_2^1}{\partial x}\frac{\partial N_1^1}{\partial x} dx = -3 \qquad K_{22}^1 = \int_0^{1/3} \frac{\partial N_2^1}{\partial x}\frac{\partial N_2^1}{\partial x} dx = 3$$

Element 2

$$K_{11}^2 = \int_{1/3}^{2/3} \frac{\partial N_1^2}{\partial x}\frac{\partial N_1^2}{\partial x} dx = 3 \qquad K_{12}^2 = \int_{1/3}^{2/3} \frac{\partial N_1^2}{\partial x}\frac{\partial N_2^2}{\partial x} dx = -3$$

$$K_{21}^2 = \int_{1/3}^{2/3} \frac{\partial N_2^2}{\partial x}\frac{\partial N_1^2}{\partial x}dx = -3 \qquad K_{22}^2 = \int_{1/3}^{2/3} \frac{\partial N_2^2}{\partial x}\frac{\partial N_2^2}{\partial x}dx = 3$$

Element 3

$$K_{11}^3 = \int_{2/3}^{1} \frac{\partial N_1^3}{\partial x}\frac{\partial N_1^3}{\partial x}dx = 3 \qquad K_{12}^3 = \int_{2/3}^{1} \frac{\partial N_1^3}{\partial x}\frac{\partial N_2^3}{\partial x}dx = -3$$

$$K_{21}^3 = \int_{2/3}^{1} \frac{\partial N_2^3}{\partial x}\frac{\partial N_1^3}{\partial x}dx = -3 \qquad K_{22}^3 = \int_{2/3}^{1} \frac{\partial N_2^3}{\partial x}\frac{\partial N_2^3}{\partial x}dx = 3$$

Calculating different F_i^e

$$F_1^1 = \int_0^{1/3} N_1^1 f dx = \int_0^{1/3} \frac{1/3 - x}{1/3} sin(x)dx = 1 - 3sin(1/3) = 1.841590961e-002$$

$$F_2^1 = \int_0^{1/3} N_2^1 f dx = \int_0^{1/3} \frac{x - 0}{1/3} sin(x)dx = 3sin(1/3) - cos(1/3) = 3.662714407e-002$$

$$F_1^2 = \int_{1/3}^{2/3} N_1^2 f dx = \int_{1/3}^{2/3} \frac{2/3 - x}{1/3} sin(x)dx = cos(1/3) - 3sin(2/3) + 3sin(1/3) = 7.143162749e-002$$

$$F_2^2 = \int_{1/3}^{2/3} N_2^2 f dx = \int_{1/3}^{2/3} \frac{x - 1/3}{1/3} sin(x)dx = 3sin(2/3) - cos(2/3) - 3sin(1/3) = 8.763805804e-002$$

$$F_1^3 = \int_{2/3}^{1} N_1^3 f dx = \int_{2/3}^{1+} \frac{2/3 - x}{1/3} sin(x)dx = cos(2/3) - 3sin(1) + 3sin(2/3) = 1.165837156e-001$$

$$F_2^3 = \int_{2/3}^{1} N_2^3 f dx = \int_{2/3}^{1} \frac{x - 2/3}{1/3} sin(x)dx = 3sin(1) - cos(1) - 3sin(2/3) = 1.290012393e-001$$

The matrix becomes,

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.841590961e-002 - Q_1 \\ 3.662714407e-002 + 7.143162749e-002 \\ 8.763805804e-002 + 1.165837156e-001 \\ 1.290012393e-001 + Q_4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.841590961e-002 - Q_1 \\ 1.080587716e-001 \\ 2.042217736e-001 \\ 1.290012393e-001 + Q_4 \end{bmatrix}$$

$$
\begin{aligned}
-3u_2 &= 1.841590961e-002 - Q_1 && (1)\\
6u_2 - 3u_3 &= 1.080587716e-001 && (2)\\
-3u_2 + 6u_3 - 9 &= 2.042217736e-001 && (3)\\
-3u_3 + 9 &= 1.290012393e-001 + Q_4 && (4)
\end{aligned}
$$

Solving, equation (2) and (3), we get

$$u_3 = 2.045382616$$

$$u_2 = 1.040701103$$

$$Q_1 = 3.158529015$$

$$Q_4 = 2.734850913$$

Actual Solutions from analytic equation,

$$u_3 = 2.057389147$$

$$u_2 = 1.046704369$$

$$Q_1 = 3.158529015$$

$$Q_4 = 2.698831321$$

Absolute Percentage Error in values,

$$
\begin{aligned}
Error(u_3) &: 0.58\%\\
Error(u_2) &: 0.57\%\\
Error(Q_1) &: 0\%\\
Error(Q_4) &: 1.32\%
\end{aligned}
$$