Finite Elements - Homework 2: Plane Elasticity

We have a triangular thin plate that is deformed under its self weight and an impose vertical displacement δ on the tip:

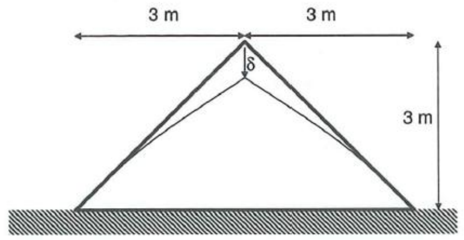

Figure 1: Geometry of the structure and prescribed displacement δ

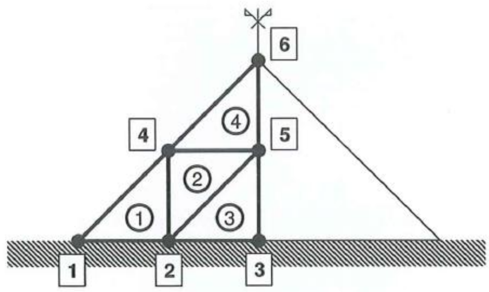

Figure 2: Mesh description. Node numbers are in squared boxes and element numbers in circles.

We use a plane stress model to analyze the structural response of the plate and using the symmetry of the problem only the left half of the domain is analyzed. We have:

- Thickness = t = 1m

- E = 10 GPa

- v = 0.2

- δ = 10^{-2} m

- ρg = 10^3 N/m^2

Part 1:

The strong form of this problem consist on the governing differential equation, the constitutive equation and the boundary conditions.
The differential equation, for a steady state problem:

$$\nabla \cdot \sigma + \rho b = 0$$

and the relation between stress and strain is:

$$\sigma = D\varepsilon$$

This exercise consist on a plane model so the constitutive matrix is like that:

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

And as we know the components of the matrix take the following form because its a plane stress problem and an isotropic material (we have the same Young modulus and Poisson ratio in every direction).

$$d_{11} = d_{22} = \frac{E}{1-\nu^2}$$
$$d_{12} = d_{21} = \nu d_{11}$$
$$d_{33} = \frac{E}{2(1+\nu)} = G$$

We can obtain the strain doing that:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

And we have to impose the boundary conditions. In this problem the BC are:

- At the bottom: u1=u2=u3 = 0 ; v1=v2=v3=0
- ux = 0 at x=3 if we take the origin of coordinates at the left vertex of the bottom. So u5=u6=0
- v6=0.01 m at the top

Part 2:

Describe the mesh shown in figure 2 by giving the arrays of nodal coordinates X and the connectivity matrix T. In order to simplify the computations select the local numbering of nodes such that, in every element, the node in the right angle vertex has local number equal to 1

The coordinates are given by:

$$X = \begin{pmatrix} 0 & 0 \\ 1.5000 & 0 \\ 3.0000 & 0 \\ 1.5000 & 1.5000 \\ 3.0000 & 1.5000 \\ 3.0000 & 3.0000 \end{pmatrix}$$

Numeration:

Element 1: Node 2: 1 , Node 4: 2 , Node1 : 3 (numeration from global to local)

Element 2: Node 4: 1, Node 2: 2, Node 5: 3

Element 3: Node 3: 1, Node 5: 2, Node 2: 3

Element 4: Node 5: 1, Node 6: 2, Node 4: 3

so the connectivity matrix is:

$$T = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{pmatrix}$$

As we have said the boundary conditions because is a symmetric problem under a vertical force:

u = 0

v1=v2=v3=0 (embedding)

v4 is different of 0, v5 is different of 0

and v6=δ

so we have:

u=0

$v(x,y) = N_1(x,y)v_1 + N_2(x,y)v_2 + N_3(x,y)v_3$

Part 3

Now we start calculating the shape functions for every element. We put the origin of coordinates at global node 1 (in the left vertex of the triangle).

We know that:

(x_i, y_j) ,con $i, j = 1,2,3,$ las coordenadas de los nodos
(u_i, v_j) ,con $i, j = 1,2,3,$ los desplazamientos de los nodos

N_i, con $i, j = 1,2,3,$ las funciones de forma:

$$N_1(x,y) = [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] / 2A_e$$
$$N_2(x,y) = [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] / 2A_e$$
$$N_3(x,y) = [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] / 2A_e$$
siendo $2A_e = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$

and applying this:

Element 1

x1=1.5 , y1=0 ; x2=1.5, y2=1.5 ; x3=0 , y3=0.

$2A_e$=2.25

N1= $\frac{1}{2Ae}(1.5x-1.5y)$

N2= $\frac{1}{2Ae}(1.5y)$

N3= $\frac{1}{2Ae}(2.25-1.5x)$

v = $\frac{1}{2Ae}\cdot[(1.5x-1.5y)\cdot v1+(1.5y)\cdot v2+(2.25-1.5x)\cdot v3]$

Element 2

x1=1.5, y1=1.5 ; x2=1.5, y2=0 ; x3=3, y3= 1.5

N1= $\frac{1}{2Ae}(2.25-1.5x+1.5y)$

N2= $\frac{1}{2Ae}(2.25-1.5y)$

N3= $\frac{1}{2Ae}(-2.25+1.5x)$

v = $\frac{1}{2Ae}\cdot[(2.25-1.5x+1.5y)\cdot v1+(2.25-1.5y)\cdot v2+(-2.25+1.5x)\cdot v3]$

Element 3

x1=3, y1=0 ; x2=3,y2=1.5 ; x3=1.5, y3=0

N1= $\frac{1}{2Ae}(-2.25+1.5x-1.5y)$

N2= $\frac{1}{2Ae}(1.5y)$

N3= $\frac{1}{2Ae}(4.5-1.5x)$

v = $\frac{1}{2Ae}\cdot[(-2.25+1.5x-1.5y)\cdot v1+(1.5y)\cdot v2+(4.5-1.5x)\cdot v3]$

Element 4

x1=3, y1=1.5 ; x2=3, y2=3 ; x3=1.5, y3=1.5

N1= $\frac{1}{2Ae}(1.5x-1.5y)$

N2= $\frac{1}{2Ae}(-2.25+1.5y)$

N3= $\frac{1}{2Ae}(4.5 - 1.5x)$

v = $\frac{1}{2Ae} \cdot [(1.5x - 1.5y) \cdot v1 + (-2.25 + 1.5y) \cdot v2 + (4.5 - 1.5x) \cdot v3]$

And now we compute the strain-stress. We know that:

$\varepsilon_x = \frac{\delta u}{\delta x} = \frac{\delta N1}{\delta x}u1 + \frac{\delta N2}{\delta x}u2 + \frac{\delta N3}{\delta x}u3$

and the same for ε_y

$\gamma_{xy} = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x}$

in matrix form we have:

$$\boldsymbol{\varepsilon} = \left\{ \begin{matrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{matrix} \right\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \vdots & \frac{\partial N_2}{\partial x} & 0 & \vdots & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \vdots & 0 & \frac{\partial N_2}{\partial y} & \vdots & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \vdots & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \vdots & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \left\{ \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \right\}$$

or

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}^{(e)}$$

where

$$\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3]$$

is the element strain matrix, and

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & 0 & \vdots & b_2 & 0 & \vdots & b_3 & 0 \\ 0 & c_1 & \vdots & 0 & c_2 & \vdots & 0 & c_3 \\ c_1 & b_1 & \vdots & c_2 & b_2 & \vdots & c_3 & b_3 \end{bmatrix}$$

$$\mathbf{B}_i = \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}$$

We compute B_i for every element and we have:

Element 1

B1= $\frac{1}{2Ae}\begin{pmatrix} 1.5 & 0 \\ 0 & -1.5 \\ -1.5 & 1.5 \end{pmatrix}$ B2= $\frac{1}{2Ae}\begin{pmatrix} 0 & 0 \\ 0 & 1.5 \\ 1.5 & 0 \end{pmatrix}$ B3= $\frac{1}{2Ae}\begin{pmatrix} -1.5 & 0 \\ 0 & 0 \\ 0 & -1.5 \end{pmatrix}$

Element 2

B1= $\frac{1}{2Ae}\begin{pmatrix} -1.5 & 0 \\ 0 & 1.5 \\ 1.5 & -1.5 \end{pmatrix}$ B2 = $\frac{1}{2Ae}\begin{pmatrix} 0 & 0 \\ 0 & -1.5 \\ -1.5 & 0 \end{pmatrix}$ B3= $\frac{1}{2Ae}\begin{pmatrix} 1.5 & 0 \\ 0 & 0 \\ 0 & 1.5 \end{pmatrix}$

Element 3

B1= $\frac{1}{2Ae}\begin{pmatrix} 1.5 & 0 \\ 0 & -1.5 \\ -1.5 & 1.5 \end{pmatrix}$ B2= $\frac{1}{2Ae}\begin{pmatrix} 0 & 0 \\ 0 & 1.5 \\ 1.5 & 0 \end{pmatrix}$ B3= $\frac{1}{2Ae}\begin{pmatrix} -1.5 & 0 \\ 0 & 0 \\ 0 & -1.5 \end{pmatrix}$

Element 4

B1= $\frac{1}{2Ae}\begin{pmatrix} 1.5 & 0 \\ 0 & -1.5 \\ -1.5 & 1.5 \end{pmatrix}$ B2= $\frac{1}{2Ae}\begin{pmatrix} 0 & 0 \\ 0 & 1.5 \\ 1.5 & 0 \end{pmatrix}$ B3= $\frac{1}{2Ae}\begin{pmatrix} -1.5 & 0 \\ 0 & 0 \\ 0 & -1.5 \end{pmatrix}$

As we know the stress field is:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBa}^{(e)}$$

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) + \boldsymbol{\sigma}^0 = \mathbf{DB}\,\mathbf{a}^{(e)} - \mathbf{D}\boldsymbol{\varepsilon}^0 + \boldsymbol{\sigma}^0$$

and D in plane stress is:

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

Plane stress

$$d_{11} = d_{22} = \frac{E}{1-\nu^2}$$

$$d_{12} = d_{21} = \nu d_{11}$$

$$d_{33} = \frac{E}{2(1+\nu)} = G$$

so the value of our D matrix is:

$$D = \begin{pmatrix} 10.4167e+009 & 2.0833e+009 & 0.0000e+000 \\ 2.0833e+009 & 10.4167e+009 & 0.0000e+000 \\ 0.0000e+000 & 0.0000e+000 & 4.1667e+009 \end{pmatrix}$$

<u>Stiffnes matrix</u>

$$\mathbf{K}^{(e)} = \iint_{A^{(e)}} \begin{Bmatrix} \mathbf{B}_1^T \\ \mathbf{B}_2^T \\ \mathbf{B}_3^T \end{Bmatrix} \mathbf{D}\, [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3] t\, dA = \iint_{A^{(e)}} \begin{bmatrix} \mathbf{B}_1^T\mathbf{D}\mathbf{B}_1 & \mathbf{B}_1^T\mathbf{D}\mathbf{B}_2 & \mathbf{B}_1^T\mathbf{D}\mathbf{B}_3 \\ \ddots & \mathbf{B}_2^T\mathbf{D}\mathbf{B}_2 & \mathbf{B}_2^T\mathbf{D}\mathbf{B}_3 \\ \text{Sym.} & \ddots & \mathbf{B}_3^T\mathbf{D}\mathbf{B}_3 \end{bmatrix} t\, dA$$

A typical element stiffness submatrix, K_{ij}^{e} linking nodes I and j can be obtained as:

$$\mathbf{K}_{ij}^{(e)} = \iint_{A^{(e)}} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j t\, dA$$

$$\mathbf{K}_{ij}^{(e)} = \iint_{A^{(e)}} \frac{1}{2A^{(e)}} \begin{bmatrix} b_j & 0 & c_j \\ 0 & c_j & b_j \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \frac{1}{2A^{(e)}} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_i & b_i \end{bmatrix} t\, dA$$

$$\mathbf{K}_{ij}^{(e)} = \left(\frac{t}{4A}\right)^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

So we compute that for every element using this formula and we obtain k11, k12, k13, k21…

Compute the stiffness matrix for each element and we have:

Ke1=

$$\begin{pmatrix} 7.2917 & -3.1250 & -2.0833 & 1.0417 & -5.2083 & 2.0833 \\ -3.1250 & 7.2917 & 2.0833 & -5.2083 & 1.0417 & -2.0833 \\ -2.0833 & 2.0833 & 2.0833 & 0 & 0 & -2.0833 \\ 1.0417 & -5.2083 & 0 & 5.2083 & -1.0417 & 0 \\ -5.2083 & 1.0417 & 0 & -1.0417 & 5.2083 & 0 \\ 2.0833 & -2.0833 & -2.0833 & 0 & 0 & 2.0833 \end{pmatrix} * 10^9$$

ke2=

$$\begin{pmatrix} 7.2917 & -3.1250 & -2.0833 & 1.0417 & -5.2083 & 2.0833 \\ -3.1250 & 7.2917 & 2.0833 & -5.2083 & 1.0417 & -2.0833 \\ -2.0833 & 2.0833 & 2.0833 & 0 & 0 & -2.0833 \\ 1.0417 & -5.2083 & 0 & 5.2083 & -1.0417 & 0 \\ -5.2083 & 1.0417 & 0 & -1.0417 & 5.2083 & 0 \\ 2.0833 & -2.0833 & -2.0833 & 0 & 0 & 2.0833 \end{pmatrix} *10^9$$

ke3=

$$\begin{pmatrix} 7.2917 & -3.1250 & -2.0833 & 1.0417 & -5.2083 & 2.0833 \\ -3.1250 & 7.2917 & 2.0833 & -5.2083 & 1.0417 & -2.0833 \\ -2.0833 & 2.0833 & 2.0833 & 0 & 0 & -2.0833 \\ 1.0417 & -5.2083 & 0 & 5.2083 & -1.0417 & 0 \\ -5.2083 & 1.0417 & 0 & -1.0417 & 5.2083 & 0 \\ 2.0833 & -2.0833 & -2.0833 & 0 & 0 & 2.0833 \end{pmatrix} *10^9$$

ke4=

$$\begin{pmatrix} 7.2917 & -3.1250 & -2.0833 & 1.0417 & -5.2083 & 2.0833 \\ -3.1250 & 7.2917 & 2.0833 & -5.2083 & 1.0417 & -2.0833 \\ -2.0833 & 2.0833 & 2.0833 & 0 & 0 & -2.0833 \\ 1.0417 & -5.2083 & 0 & 5.2083 & -1.0417 & 0 \\ -5.2083 & 1.0417 & 0 & -1.0417 & 5.2083 & 0 \\ 2.0833 & -2.0833 & -2.0833 & 0 & 0 & 2.0833 \end{pmatrix} *10^9$$

So the global stiffness matrix (K) is:

$$\begin{pmatrix} 0.5208 & 0 & -0.5208 & 0.1042 & 0 & 0 & 0 & -0.1042 & 0 & 0 & 0 & 0 \\ 0 & 0.2083 & 0.2083 & -0.2083 & 0 & 0 & -0.2083 & 0 & 0 & 0 & 0 & 0 \\ -0.5208 & 0.2083 & 1.4583 & -0.3125 & -0.5208 & 0.1042 & -0.4167 & 0.3125 & 0 & -0.3125 & 0 & 0 \\ 0.1042 & -0.2083 & -0.3125 & 1.4583 & 0.2083 & -0.2083 & 0.3125 & -1.0417 & -0.3125 & 0 & 0 & 0 \\ 0 & 0 & -0.5208 & 0.2083 & 0.7292 & -0.3125 & 0 & 0 & -0.2083 & 0.1042 & 0 & 0 \\ 0 & 0 & 0.1042 & -0.2083 & -0.3125 & 0.7292 & 0 & 0 & 0.2083 & -0.5208 & 0 & 0 \\ 0 & -0.2083 & -0.4167 & 0.3125 & 0 & 0 & 1.4583 & -0.3125 & -1.0417 & 0.3125 & 0 & -0.1042 \\ -0.1042 & 0 & 0.3125 & -1.0417 & 0 & 0 & -0.3125 & 1.4583 & 0.3125 & -0.4167 & -0.2083 & 0 \\ 0 & 0 & 0 & -0.3125 & -0.2083 & 0.2083 & -1.0417 & 0.3125 & 1.4583 & -0.3125 & -0.2083 & 0.1042 \\ 0 & 0 & -0.3125 & 0 & 0.1042 & -0.5208 & 0.3125 & -0.4167 & -0.3125 & 1.4583 & 0.2083 & -0.5208 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2083 & -0.2083 & 0.2083 & 0.2083 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1042 & 0 & 0.1042 & -0.5208 & 0 & 0.5208 \end{pmatrix} *10^10$$

And now to solve the problem we have to compute the forces and use the BC for impose the displacement.

The body forces for each node of the element were computed. As it is uniformly distributed along all the elements. Our structure is under its self weight so we only have:

$$f_i = \frac{(At)^{(e)}}{3} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix}$$

As all of the element have the same are the force vector is the same for each one of them:

Local force (fl) =

$$\begin{pmatrix} 0 \\ -375 \\ 0 \\ -375 \\ 0 \\ -375 \end{pmatrix}$$

Using the connectivity matrix we can do the assembly of the global force vector, and we have the global force vector (fg):

fg=

$$\begin{pmatrix} 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -375 \end{pmatrix}$$

We have a system like:

K·u = f

Now we impose the BC that we Know so we can cross out the rows and columns that are 0. And after that we have:

$$K_{red} = 1*10^9 * \begin{pmatrix} 1.4583 & -0.3125 & 0.3125 \\ -0.3125 & 1.4583 & -0.4167 \\ 0.3125 & -0.4167 & 1.4583 \end{pmatrix}$$

$$f_{red} = 1*10^7 * \begin{pmatrix} -1.0417 \\ -0.0001 \\ -5.2084 \end{pmatrix}$$

Solving the system we found the unknown displacements that are:

$$\begin{pmatrix} -0.0001 \\ -0.0011 \\ -0.0039 \end{pmatrix}$$

what mean that u4=-0.0001m, v4=-0.0011m and v5=-0.0039 m

so the final vector of displacements is:

u=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.0001 \\ -0.0011 \\ 0 \\ -0.0039 \\ 0 \\ -0.0100 \end{pmatrix} \text{ (units in m)}$$

We can compute the reactions and we have:

R=

$$1*10\hat{}7* \begin{pmatrix} 0.1180 \\ 0.0267 \\ 0.9081 \\ 1.1398 \\ -0.4029 \\ 2.0144 \\ -0.0000 \\ 0.0000 \\ -0.0534 \\ -0.0000 \\ -0.5698 \\ -3.1806 \end{pmatrix} \text{ (units in Newtons)}$$