Finite Elements - HW 1 Pablo Pajarón Santos

1.-

$$-u'' = f \longrightarrow \frac{d}{dx}\left(\frac{du}{dx}\right) + f = 0 \qquad 0 < x < 1$$

The integration form is obtained by multiplying this equation with test function and integrating over the domain [0,1].

$$\int_0^1 w \left(\frac{d}{dx}\left(\frac{du}{dx}\right) + f \right) dx = 0$$

Using integration by parts:

$$\int_0^1 w \cdot f \, dx + \int_0^1 w \left(\frac{d}{dx}\left(\frac{du}{dx}\right)\right) dx = 0 \qquad \begin{cases} u = w & du = w' \\ v = \frac{du}{dx} & dv = \frac{d\left(\frac{du}{dx}\right)}{dx} = u'' = +\frac{dw}{dx} \end{cases}$$

$$\int_0^1 w \cdot f \, dx = +\int_0^1 \frac{du}{dx}\frac{dw}{dx} dx - \left[w \frac{du}{dx} \right]_0^1$$

Discretization

$$\int_0^1 w_i f_i \, dx = \int_0^1 \frac{dw_i}{dx} \sum_{j=0}^{n} \frac{dN_j}{dx} u_j \, dx - \left[w_i \frac{du}{dx} \right]_0^1$$

Appying Galerkin $\rightarrow$ $(w_i = N_i)$ $\longrightarrow$ $\left[w_i \frac{du}{dx} \right]_0^1 = \left[N_i \frac{du}{dx} \right]_0^1 = \underbrace{N_1 u'\Big|_{x=1}}_{-q_1} - \underbrace{N_0 u'\Big|_{x=0}}_{q_0}$

$$\int_0^1 N_i f_i \, dx = \int_0^1 \frac{dN_i}{dx} \sum_{j=0}^{n} \frac{dN_j}{dx} u_j \, dx + q_1 - q_0$$

$$\sum_0^{n_e} \int_{l_e} N_i f_i \, dx + q_0 - q_1 = \sum_0^{n_e} \int_{l_e} \frac{dN_i}{dx} \sum_{j=0}^{n} \frac{dN_j}{dx} u_j \, dx$$

The approximation of $u(x)$ is:

$$u(x) = u^h(x) = N_0(x) u_0 + N_1(x) u_1 + \dots + N_n(x) u_n$$

$$= 0 . N + \sum_{i=0}^{n-1} N_i(x) u_i + \alpha N_n$$

2.-

The linear system of equations that must be solved is:

$$\boxed{K \cdot u = f}$$

Where the shape function for each element are:

$$\begin{cases} N_1^{(e)}(x) = \dfrac{x_2^{(e)} - x}{\ell^{(e)}} \\ N_2^{(e)}(x) = \dfrac{x - x_1^{(e)}}{\ell^{(e)}} \end{cases}$$

The stiffness matrix element are:

$$K_{ij}^{(e)} = \int_{(e)} \frac{dN_i}{dx} \cdot \frac{dN_j}{dx}\, dx$$

$$\begin{cases} K_{11}^{(e)} = \displaystyle\int_{(e)} \frac{-1}{l_{(e)}} \cdot \frac{-1}{l_{(e)}}\, dx = \frac{1}{l_{(e)}} \\ K_{12}^{(e)} = K_{21}^{(e)} = \displaystyle\int_{e} \frac{-1}{l_{(e)}} \frac{1}{l_{(e)}}\, dx = -\frac{1}{l_{(e)}} \\ K_{22}^{(e)} = \displaystyle\int_{(e)} \frac{1}{l_{(e)}} \cdot \frac{1}{l_{(e)}}\, dx = \frac{1}{l_{(e)}} \end{cases}$$

Global stiffness matrix in 1D

$$[K] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & & & \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & & & \\ & K_{21}^{(2)} & K_{22}^{(2)} + \dots & \dots + K_{11}^{(n-1)} & K_{12}^{(n-1)} & \\ & & & K_{21}^{(n-1)} & K_{22}^{(n-1)} + K_{11}^{(n)} & K_{12}^{(n)} \\ & & & & K_{21}^{(n)} & K_{22}^{(n)} \end{bmatrix}$$

Nodal force vector elements are:

$$f_i = \int_{l_e} N_i f_i + f|_{x=0} - f_1|_{x=1}$$

And the global matrix

$$[f] = \begin{bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ \vdots \\ f_2^{(n-1)} + f_1^{(n)} \\ f_2^{(n)} - q_1 \end{bmatrix}$$

3.-

There are 3 elements, $n=3$ length is equal to $\frac{1}{3}$

$f(x) = \sin x$

$\alpha = 3$

Now the problem is: $\begin{cases} -u'' = \sin x & \text{in }]0,1[\\ u(0) = 0 \\ u(1) = 3 \end{cases} \longrightarrow [u] = \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ 3 \end{bmatrix}$

Stiffness matrix is:

$$\begin{cases} K_{11}^{(e)} = 3 \\ K_{12}^{(e)} = K_{21}^{(e)} = -3 \\ K_{22}^{(e)} = 3 \end{cases} \longrightarrow [K] = 3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The force vector in each node (1D)

$$f_1^e = \int_{x_1}^{x_2} N_1 \sin x \, dx = \int_{x_1}^{x_2} \left(\frac{x_2 - x}{l}\right) \sin x \, dx = -\int_{x_1}^{x_2} +\left(\frac{\cos x}{l}\right) dx + \left[(-\cos x) \cdot \cancel{\frac{x_2 - x}{l}}\right]_{x_1}^{x_2}$$

0 for x_2

1 for x_1

$$f_1^e = -\frac{\sin x_2}{l} + \frac{\sin x_1}{l} + \cos x_1$$

$$f_2^e = \int_{x_1}^{x} \frac{(x - x_1)}{l} \sin x \, dx = -\int_{x_1}^{x_2} -\left(\frac{\cos x}{l}\right) dx + \left[(-\cos x) \cancel{\frac{x - x_1}{l}}\right] = \int_{x_1}^{x_2} N_2 \sin x \, dx$$

0 for x_1

1 for x_2

$$f_2^e = \frac{\sin x_2}{l} - \frac{\sin x_1}{l} - \cos x_2$$

where x_1 and x_2 are the interval points in each element $\begin{cases} e_1 = [0, \frac{1}{3}] \\ e_2 = [\frac{1}{3}, \frac{2}{3}] \\ e_3 = [\frac{2}{3}, 1] \end{cases}$

l: length $= \frac{1}{3}$

$$f_1^{(1)} = -3\sin\frac{1}{3} + 1 = 0{,}018$$

$$f_2^{(1)} = 3\sin\frac{1}{3} - \cos\frac{1}{3} = 0{,}0366$$

$$f_1^{(2)} = -3\sin\frac{2}{3} + 3\sin\frac{1}{3} + \cos\frac{1}{3} = 0{,}0714$$

$$f_2^{(2)} = 3\sin\frac{2}{3} - 3\sin\frac{1}{3} - \cos\frac{2}{3} = 0{,}0876$$

$$f_1^{(3)} = -3\sin 1 + 3\sin\frac{2}{3} + \cos\frac{2}{3} = 0{,}1166$$

$$f_2^{(3)} = 3\sin 1 - 3\sin\frac{2}{3} - \cos 1 = 0{,}1290$$

The global force vector is:

$$[f] = \begin{bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} - q_1 \end{bmatrix} = \begin{bmatrix} 0{,}018 + q_0 \\ 0{,}108 \\ 0{,}2042 \\ 0{,}129 - q_1 \end{bmatrix}$$

Solving the system $[K \cdot u = f]$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0{,}018 + q_0 \\ 0{,}108 \\ 0{,}2042 \\ 0{,}129 - q_1 \end{bmatrix} \longrightarrow$$

Solving:

$$\left.\begin{array}{l} 6u_1 - 3u_2 = 0{,}108 \\ -3u_1 + 6u_2 - 9 = 0{,}2042 \end{array}\right\} \begin{array}{l} u_1 = 1{,}047 \\ u_2 = 2{,}057 \end{array}$$

$$-3u_1 = 0{,}018 + q_0 \rightarrow q_0 = -3{,}123$$

$$-3u_2 + 9 = 0{,}129 - q_1 \rightarrow q_1 = 2{,}7$$

Solving the exact solution:

$$\begin{cases} u(1/3) = \sin\frac{1}{3} + (3 - \sin 1)\frac{1}{3} = 1{,}047 = u_1 \\ u(2/3) = \sin\frac{2}{3} + (3 - \sin 1)\frac{2}{3} = 2{,}057 = u_2 \end{cases}$$

It can be proved that solution is exact at the nodes.