Plane elasticity

FINITE ELEMENTS Homework # 2

Christian Rossi

A triangular thin plate is deformed under its self-weight and an imposed vertical displacement δ on the tip. A plane stress model is used to analyze the structural response of the plate. The thickness is assumed to be equal to 1, i.e. t = 1 m. Using the symmetry of the problem, only the left half of the domain is analyzed.

The finite element discretization with a mesh of 4 3-noded linear triangular elements and 6 nodes is shown in figure 2.

1. Describe the strong form of the problem in the reduced domain (let half). Indicate accurately the BC in every edge.
2. Describe the mesh by giving the arrays of the nodal coordinates X and the connectivity matrix T.
3. Set up the linear system of equations corresponding to the discretization. How many degrees of freedom have the system to be solved?
4. Compute the FE approximation u^h.

Due to the type of the problem, which can be analyzed following the assumption of 2D elasticity plane stress problem, the strong form is:

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 \qquad \boldsymbol{x} \in \boldsymbol{\Omega}$$

Where,

$$\boldsymbol{b} = \rho \cdot \boldsymbol{g}$$

The Dirichlet boundary conditions are:

$$x\text{: } u1 = u2 = u3 = u5 = u6 = 0\, m$$

$$y\text{: } \begin{cases} u6 = 1 \cdot 10^{-2}\, m \\ u1 = u2 = u3 = 0\, m \end{cases}$$

The strain field can be computed as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\Upsilon_{xy} & 0 \\ \frac{1}{2}\Upsilon_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

Where the non-zero components of the strain tensor take the form,

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x)$$

$$\varepsilon_z = -\frac{1}{E}v(\sigma_x - \sigma_y)$$

$$\Upsilon_{xy} = \frac{1}{G}\tau_{xy}$$

And the constitutive equation for plane stress linear elasticity can be written as

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \Upsilon_{xy} \end{pmatrix}$$

Where E is the *Young Modulus* and v is the *Poisson's ratio*.

And the relation between strain and displacement is:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Table 1 and table 2 and show the nodal coordinates X and the connectivity matrix T.

Table 1:Nodal coordinates.

Node	X	Y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Table 2: Coonectivity matrix.

Element	Nodes		
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

Picture1 shows the domain decomposed in 4 elements and figure 2 shows the local numbering of the nodes for a single element: the node in the right angle vertex has local numbering equal to 1 and the numeration follows a counterclockwise criteria.

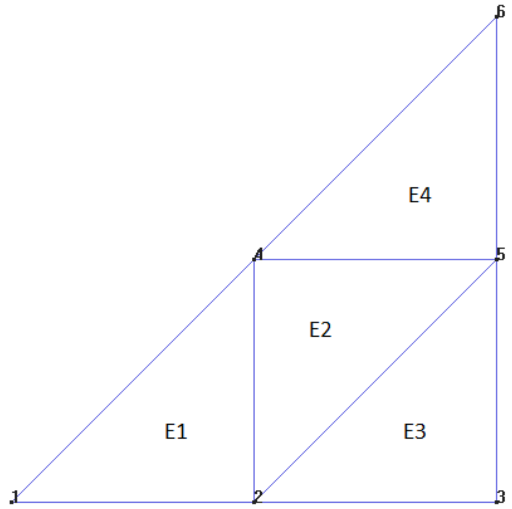

Figure 1

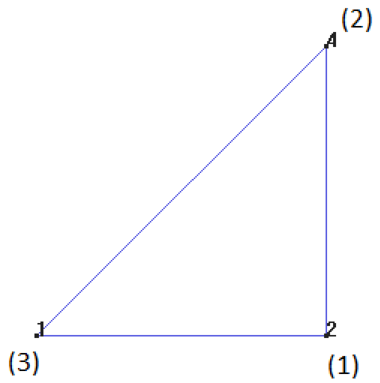

Figure 2

The first step to obtain the approximate solution is discretizing the domain as a mesh of triangular elements. The three nodes have a global numbering of *i,j,k* which corresponds to the local numbering 1,2,3. The approximate solution of the displacement in x and y directions can be expressed as:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Where N_i are the shape function and (u_i,v_i) are node displacement in x and y directions.

The equation can be written as:

$$\boldsymbol{u} = \boldsymbol{N} \cdot \boldsymbol{a}^{\boldsymbol{e}}$$

N and $\mathbf{a}^{(e)}$ contain as many matrices N_i and vectors $a^{(e)}$ as element nodes.

The expression of the shape function is found to be as:

$$N_i = \frac{1}{2 \cdot A^e}(a_i + b_i x + c_i y)$$

Where A^e is the area of the element and

$$a_i = x_j y_k - x_k y_j$$
$$b = y_l - y_k$$
$$c = x_k - x_j$$

The shape function takes the value 1 at node i and zero at the other two nodes.

Strain and stress are obtained as

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}^e$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{a}^e$$

Where **B** contains as many matrices as element nodes and is obtained as:

$$B_i = \frac{1}{2A^e}\begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}$$

And **D** is the constitutive matrix.

In this case, as the problem is a *plane stress* and the material is *isotropic*, the constitutive matrix is:

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

For isotropic elasticity plan stress problem we have,

$$d_{11} = d_{22} = \frac{E}{1 - v^2}$$

$$d_{12} = d_{21} = v d_{11}$$

$$d_{33} = \frac{E}{2(1 + v)} = G$$

Where E is the *Young Modulus* and v is the *Poisson's ratio*.

The discretized equilibrium equations for the 3-nodoed triangle will be derived by applying the PVW. In FEM the equilibrium of the forces is apply at the nodes only, so a nodal point load will be defined in order to balance the external forces and the internal forces due to the element deformation.

For each individual element, the equilibrating nodal forces are obtained as:

$$\iint_{A^e} \delta\boldsymbol{\varepsilon}^{\boldsymbol{T}}\boldsymbol{\sigma} t dA = \iint_{A^e} \delta\boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{b} t dA + \oint_{l^e} \delta\boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{t} t ds + \sum_{i=1}^{3} \delta u_i U_i + \sum_{i=1}^{3} \delta v_i V_i$$

Where U and V are the equilibrating nodal forces.

After the interpolation of the virtual displacement in terms of the nodal values, substituting in the previous equation and taking into account that the virtual displacement is arbitrary, we obtain

$$[\iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{D}\boldsymbol{B} t dA]\boldsymbol{a}^{\boldsymbol{e}} - \iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{E}\boldsymbol{\varepsilon}^{\boldsymbol{0}} t dA + \iint_{A^e} \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{\sigma}^{\boldsymbol{0}} t dA$$

$$- \iint_{A^e} \boldsymbol{N}^{\boldsymbol{T}}\boldsymbol{b} t dA \ - \oint_{l^e} \boldsymbol{N}^{\boldsymbol{T}}\boldsymbol{t} t dS = \boldsymbol{q}^{\boldsymbol{e}}$$

Where $\mathbf{q}^{\mathbf{e}}$ is the equilibrating nodal forces in terms of the nodal forces due to the element deformation (first three integrals), the body forces (second integral) and the surface traction(third integral) and $\mathbf{a}^{\mathbf{e}}$ is the nodal displacement.

The global equilibrium equation could be written as

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

Where **K** is the element stiffness matrix and it can be written for a 3 nodes triangular element as

$$\boldsymbol{K}_{\boldsymbol{ij}}^{\boldsymbol{e}} = \left(\frac{t}{4A}\right)^{e} \begin{bmatrix} b_i b_j d_{11} - c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

And the body forces equally distributed for a 3 nodes triangular element can be computed as

$$f_{bi}^{e} = \frac{(At)^{e}}{3} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

In order to set up the linear system of equations, first of all, we compute the constitutive matrix **D**

$$\boldsymbol{D} = 1 \cdot 10^{9} \begin{bmatrix} 10.417 & 2.083 & 0 \\ 2.083 & 10.417 & 0 \\ 0 & 0 & 4.167 \end{bmatrix}$$

and the element stain matrix **B** for the elements 1, 3 and 4

$$\boldsymbol{B}^{(\mathbf{1\ 3\ 4})} = \begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 1.5 \\ 0 & -1.5 & -1.5 & 1.5 & 1.5 & 0 \end{bmatrix}$$

And for the element 2

$$\boldsymbol{B}^{(\mathbf{2})} = \begin{bmatrix} 0 & 0 & 1.5 & 0 & -1.5 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 1.5 \\ -1.5 & 0 & 0 & 1.5 & 1.5 & -1.5 \end{bmatrix}$$

Then we compute the **K** matrix for each element. Following the element numbering, elements 1, 3 and 4, can join the same values, which are different from element 2.

$$K^{1\ 3\ 4} =$$

5,21E+09	0	-5,2E+09	1,04E+09	0	-1E+09
0	2,08E+09	2,08E+09	-2,1E+09	-2,1E+09	0
-5,2E+09	2,08E+09	7,29E+09	-3,1E+09	-2,1E+09	1,04E+09
1,04E+09	-2,1E+09	-3,1E+09	7,29E+09	2,08E+09	-5,2E+09
0	-2,1E+09	-2,1E+09	2,08E+09	2,08E+09	0
-1E+09	0	1,04E+09	-5,2E+09	0	5,21E+09

$$K^{2} =$$

7,29E+09	-3,1E+09	-2,1E+09	1,04E+09	-5,2E+09	2,08E+09
-3,1E+09	7,29E+09	2,08E+09	-5,2E+09	1,04E+09	-2,1E+09
-2,1E+09	2,08E+09	2,08E+09	0	0	-2,1E+09
1,04E+09	-5,2E+09	0	5,21E+09	-1E+09	0
-5,2E+09	1,04E+09	0	-1E+09	5,21E+09	0
2,08E+09	-2,1E+09	-2,1E+09	0	0	2,08E+09

Once the each element matrix was obtained, we precede to assembly the global stiffness matrix $\mathbf{K}_{\mathbf{global.}}$

In order to fill the global stiffness matrix, the first element stiffness matrix K_{1_global} will occupy.

Due to a symmetricity property of the matrix we have:

$$K_{12} = K_{21}; K_{13} = K_{31}; K_{14} = K_{41};\ K_{15} = K_{51}; K_{16} = K_{61}$$

$$K_{23} = K_{32}; K_{24} = K_{42}; K_{25} = K_{52}; K_{26} = K_{62}$$

$$K_{34} = K_{43}; K_{35} = K_{53}; K_{36} = K_{63}$$

$$K_{45} = K_{54}; K_{46} = K_{64}; K_{56} = K_{65};$$

K55	K56	K15	K16			K35	K45				
K56	K66	K25	K26			K36	K46				
K15	K16	K11	K12			K13	K14				
K25	K26	K12	K22			K23	K24				
K35	K45	K13	K14			K33	K34				
K36	K46	K23	K24			K34	K44				

The second element stiffness matrix K_{2_global} will occupy

		K33	K34			K31	K32	K35	K36		
		K34	K44			K41	K42	K45	K46		
		K31	K32			K11	K12	K15	K16		
		K41	K42			K12	K22	K25	K26		
		K35	K36			K15	K16	K55	K45		
		K45	K46			K25	K26	K45	K66		

The third element stiffness matrix K_{3_global} occupies

		K55	K56	K51	K52			KK53	K54		
		K56	K66	K61	K62			K63	K64		
		K51	K52	K11	K12			K13	K14		
		K61	K62	K12	K22			K23	K24		
		KK53	K54	K13	K14			K33	K34		
		K63	K64	K23	K24			K34	K44		

And finally the fourth element stiffness matrix K_{4_global} occupies

						K55	K56	K51	K52	K53	K54
						K56	K66	K61	K62	K63	K64
						K51	K52	K11	K12	K13	K14
						K61	K62	K12	K22	K23	K24
						K53	K54	K13	K14	K33	K34
						K63	K64	K23	K24	K34	K44

And the global vector body forces will be computed as:

$$f_{global} = (0, f_b, 0, 3f_b, 0, f_b, 0, 3f_b, 0, 3f_b, 0, f_b)^T$$

The global stiffness matrix it will be built as

$$K_{global} = \sum_{i=1}^{4} K_{i_{global}}$$

Where K_{global} is

5,21E+09	0	-5,2E+09	1,04E+09	0	0	0	-1E+09	0	0	0	0
0	2,08E+09	2,08E+09	-2,1E+09	0	0	-2,1E+09	0	0	0	0	0
-5,2E+09	2,08E+09	1,46E+10	-3,1E+09	-5,2E+09	1,04E+09	-4,2E+09	3,13E+09	0	-3,1E+09	0	0
1,04E+09	-2,1E+09	-3,1E+09	1,46E+10	2,08E+09	-2,1E+09	3,13E+09	-1E+10	-3,1E+09	0	0	0
0	0	-5,2E+09	2,08E+09	7,29E+09	-3,1E+09	0	0	-2,1E+09	1,04E+09	0	0
0	0	1,04E+09	-2,1E+09	-3,1E+09	7,29E+09	0	0	2,08E+09	-5,2E+09	0	0
0	-2,1E+09	-4,2E+09	3,13E+09	0	0	1,46E+10	-3,1E+09	-1E+10	3,13E+09	0	-1E+09
-1E+09	0	3,13E+09	-1E+10	0	0	-3,1E+09	1,46E+10	3,13E+09	-4,2E+09	-2,1E+09	0
0	0	0	-3,1E+09	-2,1E+09	2,08E+09	-1E+10	3,13E+09	1,46E+10	-3,1E+09	-2,1E+09	1,04E+09
0	0	-3,1E+09	0	1,04E+09	-5,2E+09	3,13E+09	-4,2E+09	-3,1E+09	1,46E+10	2,08E+09	-5,2E+09
0	0	0	0	0	0	0	-2,1E+09	-2,1E+09	2,08E+09	2,08E+09	0
0	0	0	0	0	0	-1E+09	0	1,04E+09	-5,2E+09	0	5,21E+09

The body forces vector, due to the body is deformed under its self-weight, has only a vertical component b_y [N].

$$f_{bi}^{e} = \frac{(9/8)}{3}\begin{pmatrix} 0 \\ -1000 \end{pmatrix}$$

The global vector body force is:

$$f_{global} = (0, -375, 0, -1125, 0, -375, 0, -1125, 0\ , -1125, 0, -375)^{T}$$

After applying the boundaries condition and simplify the global system of equation, we have the follow system:

$$1.4583e10 \cdot u_{44} - 3.125e9 \cdot u_{45} + 3.125e9 \cdot u_{55} \ = -10417e7$$

$$-3.125e9 \cdot u_{44} + 1.4583e10 \cdot u_{45} - 4.1667e9 \cdot u_{55} = -1125$$

$$3.125e9 \cdot u_{44} - -4.1667e9 \cdot u_{45} + 1.4583e10 \cdot u_{55} \ = -5.208e$$

So the linear system of equations to solve has three degrees of freedom, which are the solution at the node 4 (x and y directions) and at the node 5 (y direction).

Results

The FEM approximation gives us the following results.

Table 1: Displacement [m]

Node	x displacement	y displacement
1	0	0
2	0	0
3	0	0
4	-1.282e-4	-1.133e-3
5	0	-3.867e-3
6	0	-0.1

Table 2: Reaction forces [N]

Node	R_x	R_y
1	1.179e6	2.674e5
2	9.081e6	1.139e7
3	-4.028e6	2.014e7
4	0	0
5	-5.341e5	0
6	-5.698e6	-3.180e7

Table 5: Strains

Element	ε_x	ε_y	Υ_{xy}
1	0	-1.698e-3	-1.923e-4
2	1.923e-4	-1.698e-3	-4.294e-3
3	0	-5.801e-3	0
4	1.923e-4	-9.198e-3	-4.102e-3

Table 4: Stresses [N/m^2]

Element	σ_x	σ_y	T_{xy}
1	-3.539e6	-17.696e6	-801.282e3
2	-1.536e6	-17.296e6	-17.895e6
3	-12.086e6	-60.431e6	0
4	-17.160e6	-95.417e6	-17.094e6