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HOMEWORK 2

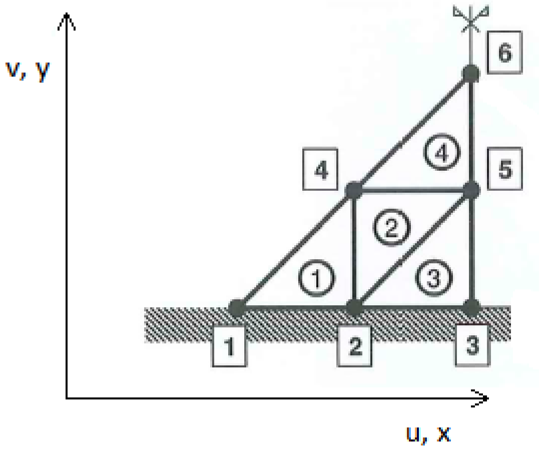
SOLUTION

1. Strong form.

The problem considers deformation of the triangle under its self weight. To describe the problem, it is necessary to use governing equations and, particularly, linear momentum balance equation which takes the form:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0,$$

where $\boldsymbol{\sigma}$ – stress tensor, ρ – density of the material, $\mathbf{b}$ – vector of body forces.



Furthermore, it is important to define boundary conditions. The displacement vector is defined as

$$\boldsymbol{u}(x,y) = \begin{cases} u(x,y) \\ v(x,y) \end{cases}$$

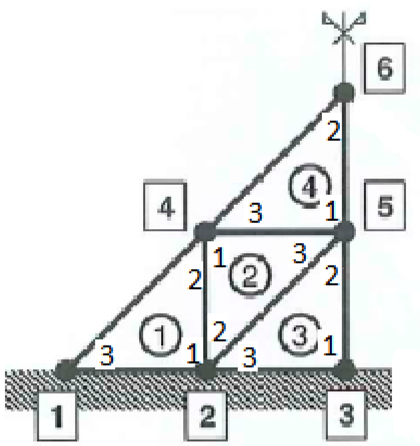
As there are no displacements in nodes 1, 2 and 3, components of the vector in these equal to 0: $u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = 0$. Also, there is no displacement in the nodes 5 and 6 along the x-axes and displacement in the node 6 along y-axes equals to -δ: $u_5 = u_6 = 0, v_6 = -\delta$.

Assuming, the strong form takes the view:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0$$

$$\boldsymbol{u_1} = \begin{cases} 0 \\ 0 \end{cases}, \boldsymbol{u_2} = \begin{cases} 0 \\ 0 \end{cases}, \boldsymbol{u_3} = \begin{cases} 0 \\ 0 \end{cases}, \boldsymbol{u_4} = \begin{cases} u_4 \\ v_4 \end{cases}, \boldsymbol{u_5} = \begin{cases} 0 \\ v_5 \end{cases}, \boldsymbol{u_6} = \begin{cases} 0 \\ -\delta \end{cases}$$

2. Let us describe mesh topology, selecting the local numbering of the nodes such that, in every element, the node in the right angle vertex has local number equal to 1:



Element	Nodal connection		
	1	**2**	**3**
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

Connectivity matrix takes the form T= $\begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix}$; arrays of nodal coordinates X= $\begin{bmatrix} 1.5 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1.5 & 1.5 & 3 \\ 1.5 & 0 & 1.5 \\ 3 & 3 & 1.5 \\ 0 & 1.5 & 0 \\ 3 & 3 & 1.5 \\ 1.5 & 3 & 1.5 \end{bmatrix}$.

3. The system of equations to be solved has the form **Ka**=**f**, where **K** – stiffness matrix, **a** – vector of displacements, **f** – vector of nodal forces. To obtain the global stiffness matrix, let us define the local ones for every element, using connectivity matrix T:

$$K^{(1)} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{bmatrix} \begin{matrix} 2 \\ 4 \\ 1 \end{matrix} \quad K^{(2)} = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{matrix} 4 \\ 2 \\ 5 \end{matrix}$$

$$\begin{matrix} 2 & 4 & 1 \end{matrix} \qquad\qquad \begin{matrix} 4 & 2 & 5 \end{matrix}$$

$$K^{(3)} = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} \end{bmatrix} \begin{matrix} 3 \\ 5 \\ 2 \end{matrix} \quad K^{(4)} = \begin{bmatrix} K_{11}^{(4)} & K_{12}^{(4)} & K_{13}^{(4)} \\ K_{21}^{(4)} & K_{22}^{(4)} & K_{23}^{(4)} \\ K_{31}^{(4)} & K_{32}^{(4)} & K_{33}^{(4)} \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 4 \end{matrix}$$

$$\begin{matrix} 3 & 5 & 2 \end{matrix} \qquad\qquad \begin{matrix} 5 & 6 & 4 \end{matrix}$$

The global stiffness matrix takes the form:

$$\boldsymbol{K} = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ & K_{11}^{(1)} + K_{22}^{(2)} + K_{33}^{(3)} & K_{31}^{(3)} & K_{12}^{(1)} + K_{21}^{(2)} & K_{23}^{(2)} + K_{32}^{(3)} & 0 \\ & & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ & & & K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{31}^{(4)} & K_{32}^{(4)} \\ & Symmetric & & & K_{33}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

Applying boundary conditions, vectors of displacements and forces became the following ones:

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \boldsymbol{u_4} \\ \boldsymbol{u_5} \\ \boldsymbol{u_6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -\delta \end{bmatrix} \quad \boldsymbol{f} = \begin{bmatrix} r_1 + f_3^{(1)} \\ r_2 + f_1^{(1)} + f_2^{(2)} + f_3^{(3)} \\ r_3 + f_1^{(3)} \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ r_5 + f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ r_6 + f_2^{(4)} \end{bmatrix}$$ where $r_i = \begin{cases} r_x^{(i)} \\ r_y^{(i)} \end{cases}$ - reaction in the i-th node, and

$r_5 = \begin{cases} r_x^{(5)} \\ 0 \end{cases}$. A global nodal force in a particular node is equal to sum of local nodal forces in this node.

In order the system has to be solved, it must have 9 degrees of freedom (d.o.f.) and there are reactions for every prescribed degree of freedom. Specifically, the nodes 1, 2, 3 and 6 have 2 d.o.f., the node 4 has no d.o.f. and the node 5 has 1 d.o.f.

As $\boldsymbol{u_1} = \boldsymbol{u_2} = \boldsymbol{u_3} = \boldsymbol{0}$ and there are 3 unknowns which should be found, the system can be reduced:

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{31}^{(4)} & K_{32}^{(4)} \\ K_{31}^{(2)} + K_{13}^{(4)} & K_{33}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} \\ K_{23}^{(4)} & K_{21}^{(4)} & K_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ r_5 + f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ r_6 + f_2^{(4)} \end{bmatrix} \qquad (*)$$

Finally, we will receive the system, consisting of 3 equations, because we have only 3 unknowns.

4. Compute the FE approximation $\boldsymbol{u^h}$. Use E = 10GPa, ν = 0.2, δ = $\boldsymbol{10^{-2}}$m, ρg = $\boldsymbol{10^3 \frac{N}{m^2}}$.

A typical element stiffness submatrix for the 3-noded triangular element can take the form:

$$K_{ij}^{(e)} = \iint_{A^{(e)}} B_i^T D B_j \, t dA$$

where $i,j = \{1,2,3\}$, t is the thickness, $A^{(e)}$ is the area of the element, B_j is the strain matrix of node j and D is the constitutive matrix such as **σ**=**Dε** where **ε** is the strain tensor. Developing the equation:

$K_{ij}^{(e)} = \iint_{A^{(e)}} \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \frac{1}{2A^{(e)}} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix} t dA$. Since this integrand id the constant, we obtain the following: $K_{ij}^{(e)} = (\frac{t}{4A^{(e)}})^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ b_j c_i d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$.

In case of isotropic elasticity and plane stress, elements of the constitutive matrix take a view:

$$d_{11} = d_{22} = \frac{E}{1 - \nu^2} = \frac{10 \cdot 10^9}{1 - 0.004} = \frac{125}{12} 10^9;$$

$$d_{12} = d_{21} = \nu d_{11} = 0.2 \cdot \frac{125}{12} 10^9 = \frac{25}{12} 10^9;$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10 \cdot 10^9}{2(1+0.2)} = \frac{25}{6} 10^9.$$

Elements of the matrix B can be defined as $b_i = y_j - y_k \;\; c_i = x_k - x_j$ where $i,j,k = \{1,2,3\}$. For every local element we will receive:

	b_1	b_2	b_3	c_1	c_2	c_3
1	1.5	0	-1.5	-1.5	1.5	0
2	-1.5	0	1.5	1.5	-1.5	0
3	1.5	0	-1.5	-1.5	1.5	0
4	1.5	0	-1.5	-1.5	1.5	0

Thus, every local stiffness matrix will be the same for every element. Therefore, we need to calculate only one. Using described above formulas, receive following components:

$$K_{11} = c\begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix} \; K_{12} = c\begin{bmatrix} -2 & 1 \\ 2 & -5 \end{bmatrix} \; K_{21} = c\begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$$

$$K_{22} = c\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \; K_{23} = c\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \; K_{32} = c\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$$

$$K_{33} = c\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \; K_{13} = c\begin{bmatrix} -5 & 2 \\ 1 & -2 \end{bmatrix} \; K_{31} = c\begin{bmatrix} -5 & 1 \\ 2 & -2 \end{bmatrix}$$

where $c = \frac{25}{24} 10^9$.

As body forces are uniformly distributed over the elements, the forces will take the form:

$$f_{b_i} = \frac{(At)^{(e)}}{3} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

And all local forces will be the same. As deformation of the triangle happens under its self weight, we have only one force which is gravity acting in the y-direction. And there are no forces acting in the x-direction. In this case $b_x = 0, b_y = -\rho g$. A local force for every node of every element takes the view:

$$f = \frac{\left(\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\right)\cdot 1}{3}\begin{bmatrix} 0 \\ -\rho g \end{bmatrix} = \frac{3}{8}\begin{bmatrix} 0 \\ -10^3 \end{bmatrix}$$

Applying received data to system of equations (*), we receive the following:

$$\frac{25}{24}10^9\begin{bmatrix} 14 & -3 & -10 & 3 & 0 & -1 \\ -3 & 14 & 3 & -4 & -2 & 0 \\ -10 & 3 & 14 & -3 & -2 & 1 \\ 3 & -4 & -3 & 14 & 2 & -5 \\ 0 & -2 & -2 & 2 & 2 & 0 \\ -1 & 0 & 1 & -5 & 0 & 5 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{9}{8}10^3 \\ r_x^{(5)} \\ -\frac{9}{8}10^3 \\ r_x^{(6)} \\ r_y^{(6)} - \frac{3}{8}10^3 \end{bmatrix}$$

Reducing the system, we receive:

$$\begin{bmatrix} 14 & -3 & 3 \\ -3 & 14 & -4 \\ 3 & -4 & 14 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -10^{-2} \\ -1.08\cdot 10^{-6} \\ -50001.08\cdot 10^{-6} \end{bmatrix}$$

Solving the system of equations, we the FE approximation:

$$u^h = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.00012809 \\ -0.00113258 \\ 0 \\ -0.00386765 \\ 0 \\ -0.02 \end{bmatrix}$$