FINITE ELEMENT METHOD

HOME WORK - 2

Solution

Plane Elasticity

Here we are analysing plane elasticity problem of prismatic bodies, assuming plane stress.

Given data:-

Thickness, $t = 1\,m$

Young's modulous, $E = 10\,GPa$

Poisson Ratio, $\nu = 0.2$

Vertical Displacement, $\delta = 10^{-2} = 0.01\,m$

Body force $= \rho g = 10^3\,N/m^2$

Strong form is written as,

$$b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad -(1)$$

$$b_x + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad -(2)$$

Boundary Conditions are through displacement of given nodes in ~~x & y~~ x & y direction

from the fig. given, it is Clear that,

$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$ (fixed nodes -1,2,3)

$u_5 = 0$ (Symmetry Condition)

$\delta = 10^{-2} = 0.01\,m = v_6$ (Given data)

Nodal Co-ordinates (x) & Connectivity Matrix (T):-

Nodes	x	y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Nodal Co-ordinates (x)

	Nodes		
Elements	1	2	3
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

T - Matrix (Connectivity Matrix)

Description of Mesh:-

from fig, we have four elements in order to makes the descritization easier, local numbering is made, Such that descritization easier local numbering is made, Such that in every element, the node in the right angle Vertex has a local number equal to 1, which is Shown in above figure

Now, to find the descritization of displacement field, we have

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The three nodes of triangular mesh defines linear displacement field which can be written as.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

After deriving the shape functions for 'u' alone We get,

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y) \quad \text{— (3)}$$

Where,

$$a_i = x_j x_k - x_k x_j ; \; b_i = y_j - y_k ; \; c_i = x_k - x_j$$

We know that the stiffness matrix is given by,

$$K^e = \iint_{A^e} B^T D B t \, dA \quad \text{— (4)}$$

Equivalent Nodal force vector,

$$f^e = f_\epsilon^e + f_\sigma^e + f_b^e + f_t^e \quad \text{— (5)}$$

$$f_\epsilon^e = \iint_{A^e} B^T D \epsilon^0 t \, dA$$

$$f_\sigma^e = \iint_{A^e} B^T D \sigma^0 t \, dA$$

$$f_b^e = \iint_{A^e} N^T b\, t\, dA$$

$$f_t^e = \iint_{A^e} N^T t\, dA$$

Eq. can also be written as.

$$⑥ \longrightarrow K_{ij}^{\,} = \left(\frac{I}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + C_i C_j d_{33} & b_i C_j d_{12} + b_j C_i d_{33} \\ C_i b_j d_{21} + b_i C_j d_{33} & b_i b_j d_{33} + C_i C_j d_{22} \end{bmatrix}$$

As we are dealing with plane stress problem, we have.

$$\sigma = D\epsilon$$

D is the constitutive matrix defined for given data as,

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$$d_{11} = d_{22} = \frac{E}{(1-V^2)} = \frac{10\ GPa}{(1-0.2^2)} = 10.417\ GPa$$

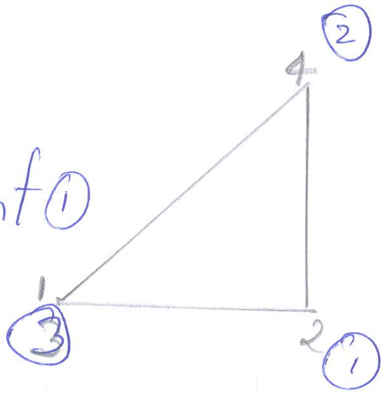
$$d_{12} = d_{21} = V d_{11} = 0.2 \times (10.417) = 2.083\ GPa.$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10}{2(1+0.2)} = 4.167 \text{ GPa}$$

To Compute Stiffness Matrix for Element 1, 3 & 4

From Nodal Co-ordinates & Considering local numbering we have,

Element ①



1, 2, 4 → Global Numbering
1, 2, 3 → Local Numbering

From fig ① : Element ①

$$(x_1, y_1)^1 = (-1.5, 0)$$

$$(x_2, y_2)^1 = (-1.5, 1.5)$$

$$(x_3, y_3)^1 = (-3, 0)$$

But,

$$b_i = y_j - y_k \quad \& \quad C_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = 1.5 \; ; \quad C_1 = x_3 - x_2 = -1.5$$

$$b_2 = y_3 - y_1 = 0 \; ; \quad C_2 = x_1 - x_3 = 1.5$$

$$b_3 = y_1 - y_2 = -1.5 \; ; \quad C_3 = x_2 - x_1 = 0$$

Now using Eq. (6) we have.

$$K_{ij} = \left(\frac{1}{4A}\right)^{e} \begin{bmatrix} b_i b_j d_{11} + C_i C_j d_{33} & b_i C_j d_{12} + b_j C_i d_{33} \\ C_i b_j d_{21} + b_i C_j d_{33} & b_i b_j d_{33} + C_i C_j d_{22} \end{bmatrix}$$

$$K_{11}^{(1)} = \frac{2}{9} \begin{bmatrix} 2.25 \times 10.417 + 2.25 \times 4.167 & -2.25 \times 2.083 - 2.25 \times 4.167 \\ -2.25 \times 2.083 + (-2.25) \times 4.167 & 2.25 \times 10.417 + 2.25 \times 4.167 \end{bmatrix}$$

$$K_{11}^{(1)} = \frac{2 \times 2.25}{9} \begin{bmatrix} 10.417 + 4.167 & -2.083 - 4.167 \\ -2.083 - 4.167 & 10.417 + 4.167 \end{bmatrix}$$

$$K_{11}^{(1)} = \begin{bmatrix} 7.292 & -3.125 \\ -3.125 & 7.292 \end{bmatrix}$$

Similarly

$$K_{12}^{(1)} = \frac{2 \times 2.25}{9} \begin{bmatrix} -4.167 & 2.083 \\ 4.167 & -10.417 \end{bmatrix} = \begin{bmatrix} -2.083 & 1.042 \\ 2.083 & -5.209 \end{bmatrix} = K_{21}^{e}$$

$$K_{13}^{(1)} = \frac{2 \times 2.25}{9} \begin{bmatrix} -10.417 & 4.167 \\ 2.083 & -4.167 \end{bmatrix} = \begin{bmatrix} -5.209 & 2.083 \\ 1.042 & -2.083 \end{bmatrix} = K_{31}^{e}$$

$$K_{22}^{(1)} = \frac{2 \times 2.25}{9} \begin{bmatrix} 4.167 & 0 \\ 0 & 10.417 \end{bmatrix} = \begin{bmatrix} 2.083 & 0 \\ 0 & 5.209 \end{bmatrix} =$$

$$K_{23}^{(1)} = \frac{2 \times 2.25}{9} \begin{bmatrix} 0 & -4.167 \\ -2.083 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.083 \\ -1.042 & 0 \end{bmatrix} = K_{32}^{e}$$

$$K_{33}^{(1)} = \frac{2 \times 2.25}{9}\begin{bmatrix} 10.417 & 0 \\ -2.083 & 4.167 \end{bmatrix} = \begin{bmatrix} 5.209 & 0 \\ 0 & 2.083 \end{bmatrix}$$

Since Nodal Co-ordinates & local Co-ordinates are same for elements (1), (3) & (4), we have

$$K_{11}^{(3)} = K_{11}^{(4)} = K_{11}^{(1)}; \quad K_{12}^{(3)} = K_{12}^{(4)} = K_{12}^{(1)}$$

$$K_{13}^{(3)} = K_{13}^{(4)} = K_{13}^{(1)}$$

$$K_{21}^{(3)} = K_{21}^{(4)} = K_{21}^{(1)}; \quad K_{22}^{(3)} = K_{22}^{(4)} = K_{22}^{(1)}$$

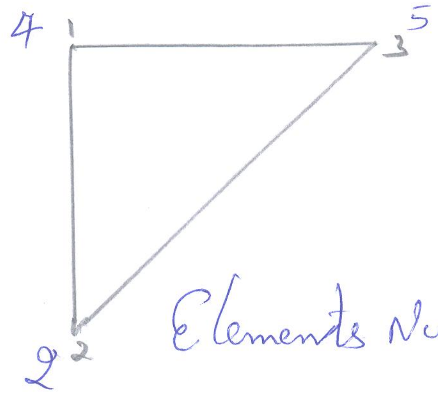
$$K_{31}^{(3)} = K_{31}^{(4)} = K_{31}^{(1)}; \quad K_{32}^{(3)} = K_{32}^{(4)} = K_{32}^{(1)}$$

$$K_{33}^{(3)} = K_{33}^{(4)} = K_{33}^{(1)}$$

Therefore elements 1, 3 & 4 have same stiffness matrix

The stiffness matrix for element (2)

From Nodal Co-ordinates and Considering local Numbering, we have.



2, 4, 5 – Global Numbering
1, 2, 3 – Local Numbering

Elements Number – 2.

$(x_1, y_1)^2 = (-1.5, 1.5)$

$(x_2, y_2)^2 = (-1.5, 0)$

$(x_3, y_3)^2 = (0, 1.5)$

We know ~~that~~ ~~table~~ that,

$$b_i = y_j - y_k \;\&\; c_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = -1.5 \;;\; c_1 = x_3 - x_2 = 1.5$$

$$b_2 = y_3 - y_1 = 0 \;;\; c_2 = x_1 - x_3 = -1.5$$

$$b_3 = y_1 - y_2 = 1.5 \;;\; c_3 = x_2 - x_1 = 0$$

Since the values are same as that of elements 1, but opp. in sign. Hence we have the same stiffness matrix for the element 2 as well.

Therefore,

$$K^1 = K^2 = K^3 = K^4$$

Assembling of Stiffness Matrix

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} = K^2 = \begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{32}^2 \end{bmatrix}$$

$$= K^3 = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{bmatrix} = K^4 = \begin{bmatrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{bmatrix}$$

The assembly of matrix is shown below:-

$$K = \begin{bmatrix} K_{33}^{(1)} & K_{13}^{1T} & 0 & K_{23}^{1} & 0 & 0 \\ 0 & K_{11}^{1}+K_{22}^{2}+K_{33}^{3} & K_{13}^{3T} & K_{12}^{1}+K_{12}^{2T} & K_{23}^{1}+K_{23}^{3T} & 0 \\ 0 & 0 & K_{11}^{3} & 0 & K_{12}^{3} & 0 \\ 0 & 0 & 0 & K_{22}^{1}+K_{11}^{2}+K_{33}^{4} & K_{13}^{2}+K_{13}^{4T} & K_{23}^{4T} \\ 0 & 0 & 0 & 0 & K_{11}^{4}+K_{22}^{3}+K_{33}^{2} & K_{12}^{4} \\ 0 & 0 & 0 & 0 & 0 & K_{22}^{4} \end{bmatrix}$$

We know that,

$$a_i^e = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \Rightarrow a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

for the given problem, the displacement matrix 'a' is given by

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

And the nodal force vector is given by

$$f = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix}$$

W.K.T.

$$\boxed{Ka = f}$$

The given problem has 12 degrees of freedom, where six of d.o.f are Constrained.

Nodal displacements

From boundary Conditions, we have

$$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0;$$

$$\text{hence } a_1 = a_2 = a_3 = 0$$

Hence we need to Consider only 4, 5 & 6 rows only from global stiffness matrix and also $u_5 = u_6 = 0$

Therefore,

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{4} & K_{13}^{2} + K_{13}^{4T} & K_{23}^{4T} \\ & K_{11}^{4} + K_{22}^{3} + K_{33}^{2} & K_{12}^{4} \\ & & K_{22}^{4} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ u_5 = 0 \\ v_5 \\ u_6 \\ v_6 = \delta = 10^{-2} \end{bmatrix} = \begin{bmatrix} f_2^1 + F_1^2 + F_3^4 \\ f_3^2 + f_2^3 + F_1^4 \\ F_2^4 \end{bmatrix} \longrightarrow (7)$$

Now substituting value of K_{ij}^e for the above matrix, we get,

$$K = \begin{bmatrix} -14.584 & -3.125 & -10.417 & 3.125 & 0 & 1.042 \\ -3.125 & 14.584 & 4.166 & -4.166 & -1.042 & 0 \\ -10.417 & 4.166 & 14.584 & -3.125 & -2.083 & 1.042 \\ 3.125 & -4.166 & -3.125 & 14.584 & 2.083 & 5.209 \\ 0 & -1.042 & -2.083 & 2.083 & 2.083 & 0 \\ 1.042 & 0 & 1.042 & -5.209 & 0 & 5.209 \end{bmatrix} GN/$$

We have from data that the whole domain deforms because of self weight with gravity acting in the direction y-axis. Therefore only body forces are significant and no surface loads.

Body forces for the equivalent nodal force is given by

$$f_{b_i} = \left(\frac{At}{3}\right)^e \begin{bmatrix} b_x \\ b_y \end{bmatrix} \longrightarrow (8)$$

but,

$$b_x = 0; \; b_y = -\rho g = -10^3$$

$$f_{b_i} = \left(\frac{2.25}{6}\right) \begin{bmatrix} 0 \\ -10^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \end{bmatrix} N$$

Substituting f_5 4K in Eq (I) we get by simplifying

$$\begin{bmatrix} 14.584 & -3.125 & 3.125 & 1.042 \\ -3.125 & 14.584 & -4.166 & 0 \\ 3.125 & -4.166 & 14.584 & -5.209 \\ 1.042 & 0 & -5.209 & 5.209 \end{bmatrix} \times 10^9 \begin{bmatrix} u_4 \\ V_4 \\ V_5 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

When we solve the above system of linear equations, we get the nodal displacement values as:

$$\boxed{\begin{aligned} U_4 &= -1.29 \times 10^{-4} \\ V_4 &= -1.13 \times 10^{-3} \\ V_5 &= -3.87 \times 10^{-3} \end{aligned}}$$