Computational Mechanics Tools
PDE Toolbox

Assignment 3

Federico Valencia Otálvaro

Master in Numerical Methods in Engineering

Universitat Politècnica de Catalunya

December 13, 2019

Contents

List of Figures

List of Tables

1 Problem Description

Given the **parabolic** partial differential equation (PDE):

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -3e^{-3t} \qquad in \quad \Omega = [0,1]^2 \tag{1}$$

With boundary conditions:

$$\frac{\partial u}{\partial n} = -y \qquad for \quad x = 0 \tag{2}$$

$$\frac{\partial u}{\partial n} = 2 + y \qquad for \quad x = 1 \tag{3}$$

$$u = x^2 + e^{-3t} \qquad for \quad y = 0 \tag{4}$$

$$\frac{\partial u}{\partial n} = x - 2 \qquad for \quad y = 1 \tag{5}$$

And initial condition:

$$u = x^2 + xy - y^2 + 1 \qquad for \quad t = 0 \tag{6}$$

With the analytical solution given by:

$$u = x^2 + xy - y^2 + e^{-3t} \tag{7}$$

The goal is, using the MATLAB PDE Toolbox:

1. Consider $t_{end} = 10$, solve the problem, and refine the initial mesh up to four times. Verify that the theoretical convergence order holds.
2. Describe how the solution is affected when the final time is modified.
3. For $t_{end} = 50$, find a more efficient manner to solve this problem. Provide numerical evidence for the new method.

2 Solution

2.1 Part 1

The analytical solution of the given PDE in the domain Ω, for $t_{end} = 10$ is:

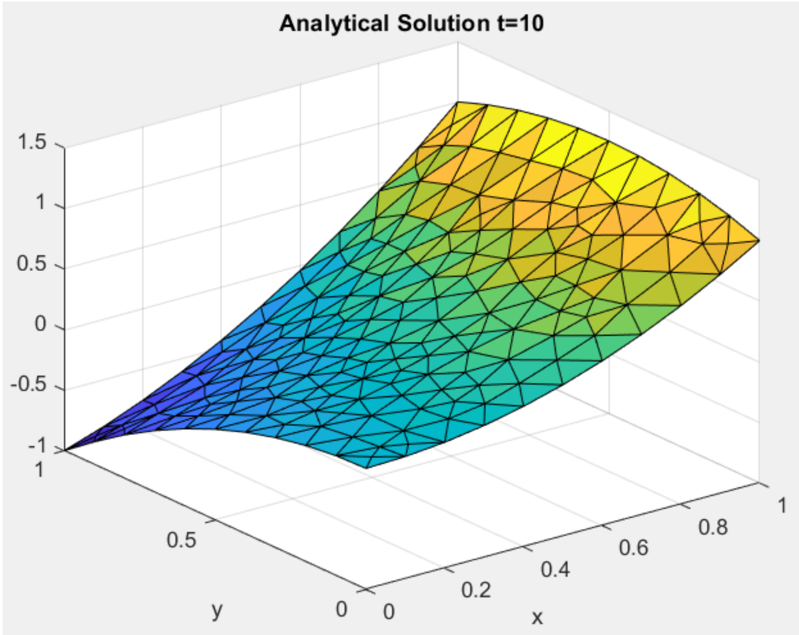

Figure 1: Analytical Solution, t=10

The MATLAB PDE Toolbox was employed to find the numerical solution for the problem through the finite element method (FEM). Initially, the domain was discretized in a mesh made up by 328 triangular elements as is shown in *Figure 2*:

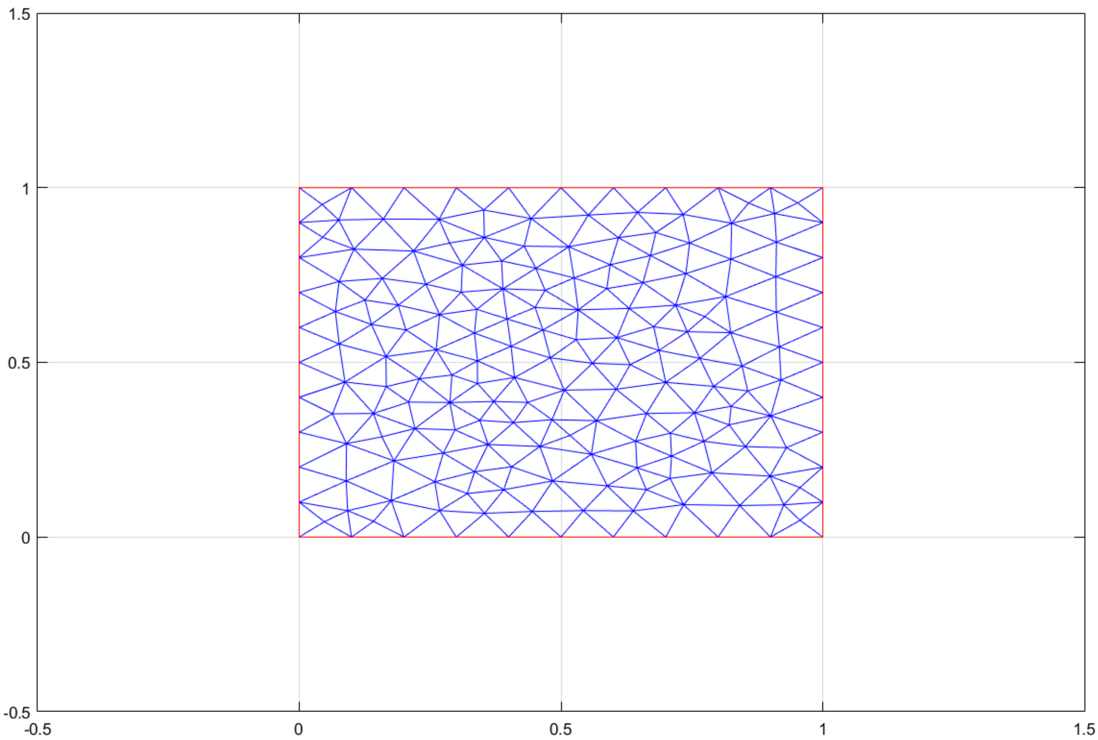

Figure 2: Initial mesh

The numerical solution obtained with this mesh, for $t_{end} = 10$ and the given boundary and initial conditions is:

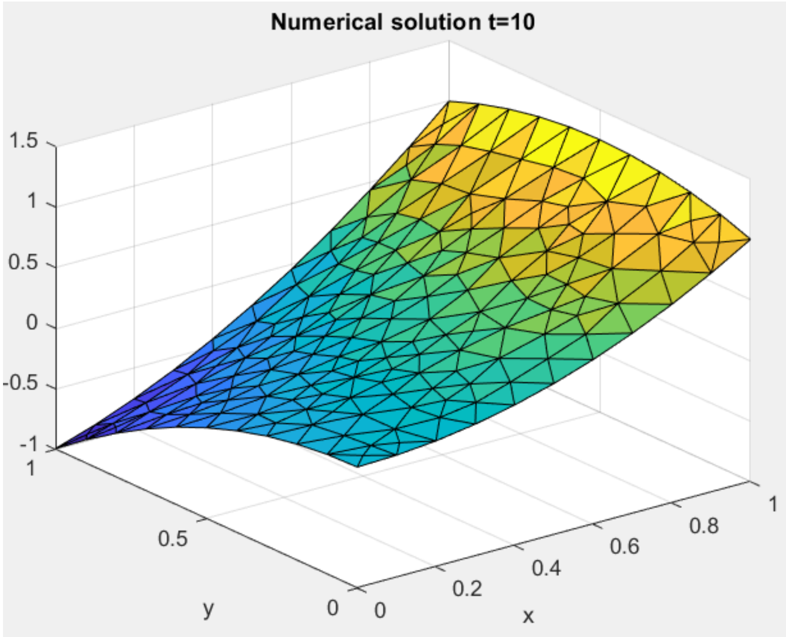

Figure 3: Numerical solution for initial mesh, t=10

Considering the error as the absolute value of the difference between the numerical and analytical solutions, it was possible to plot the error of the initial solution over the domain:

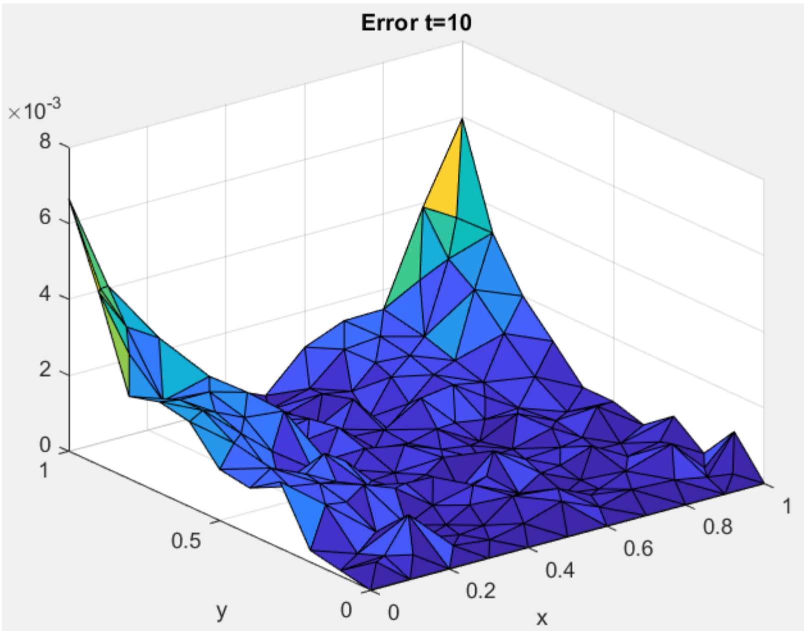

Figure 4: Error in numerical solution for initial mesh, t=10

The maximum error ($e_{max} = 0.0067$) is taken as an indicator of the quality of the solution. Now, in order to create a relation between the error and the grade of refinement of the mesh, the mesh was refined four times and the evolution of the maximum error in the solution with respect to the element size was registered. Since the area of the domain (unitary) and the number of elements are known, the average element size (height) can be defined as:

$$h = \sqrt{\frac{2}{n}} \tag{8}$$

Where:

- h: element size
- n: elements in the mesh

The obtained results are:

Mesh	Max. Error	Elements	h
Initial	0.0067	328	0.0781
Refined 1	0.0020	1312	0.0390
Refined 2	5.6787e-04	5248	0.0195
Refined 3	1.60e-04	20992	9.76e-03
Refined 4	4.44e-05	83968	4.88e-03

Table 1: Max. Error vs. Element Size, t=10

Plotting the logarithm of h against the logarithm of the maximum error, we have:

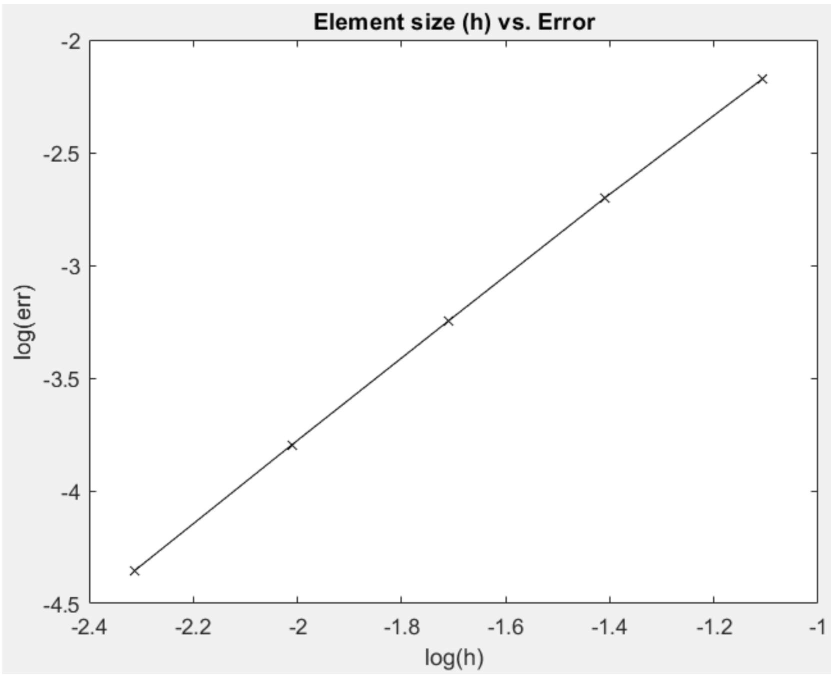

Figure 5: Max. Error vs. Element Size, t=10

The slope between two points i and $i+1$ is given by:

$$S = \frac{log_{10}(\frac{error_{i+1}}{error_i})}{log_{10}(\frac{h_{i+1}}{h_i})} \tag{9}$$

Since there are five points, it is possible to calculate four different slopes. As it is made evident in *Figure 5* that all the slopes are very similar, we calculate the mean slope.

$$S_{mean} = 1.8092 \approx 1.80 \tag{10}$$

Hence, the error is effectively reduced by decreasing the element size and the solution derived from the refined mesh is a better approximation to the analytical solution than the one derived from the initial mesh. The results for the final solution (83968 elements) are:

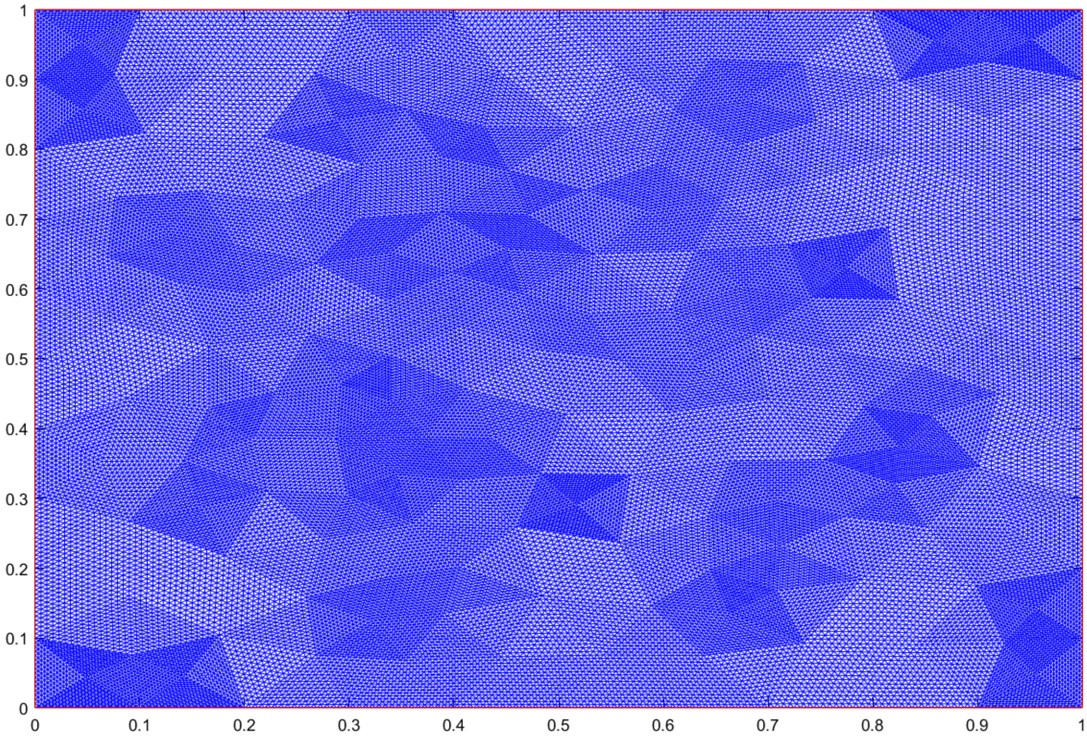

Figure 6: Refined Mesh

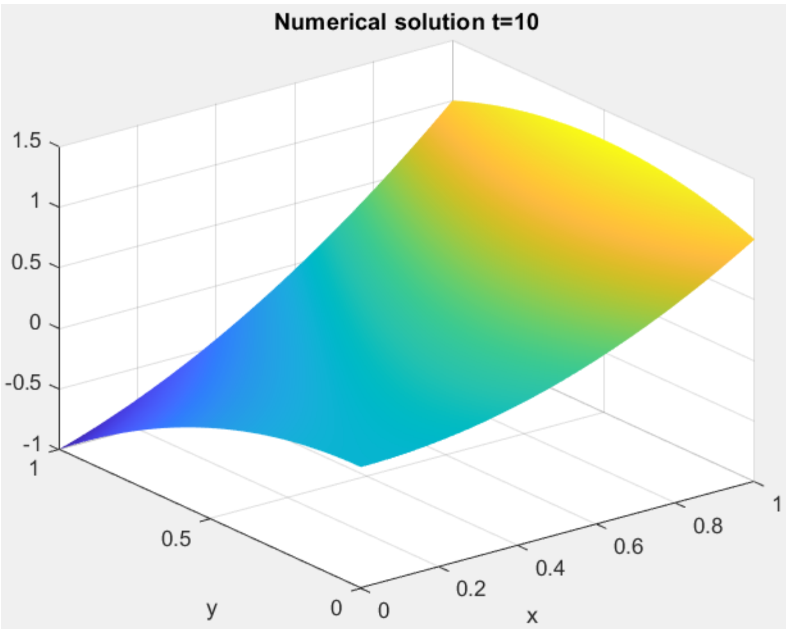

Figure 7: Numerical solution for refined mesh, t=10

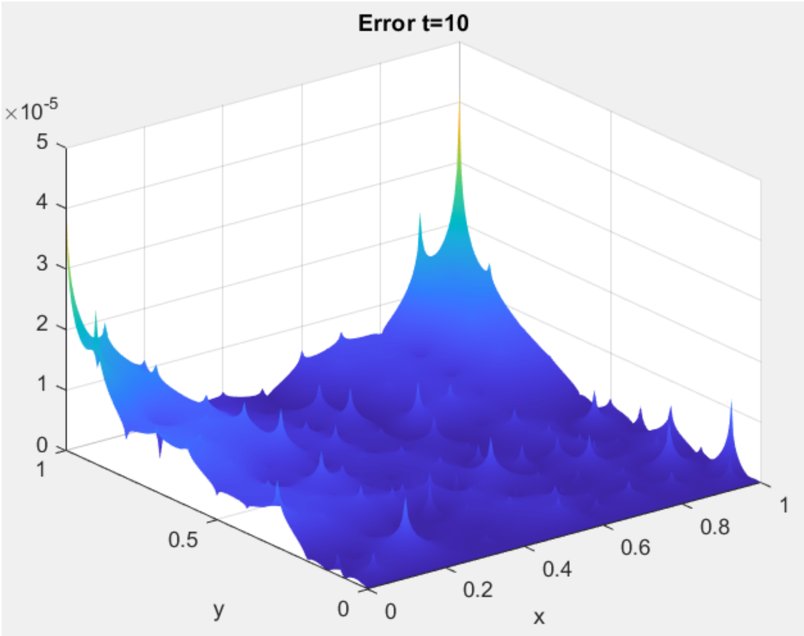

Figure 8: Error in numerical solution for refined mesh, t=10

2.2 Part 2

To study how the solution is affected by the final time, both the analytical solution and the numerical solution of the final refined mesh were calculated for times between 0 and 10, in steps of 1. The maximum error between the analytical and numerical solution for each time step was then plotted against time.

t	0	1	2	3	4	5	6	7	8	9	10
Error	0	0.6606	0.3303	0.0856	0.1078	0.0519	0.0539	0.048	0.0454	0.0446	0.0444

Table 2: Max. Error vs. Time

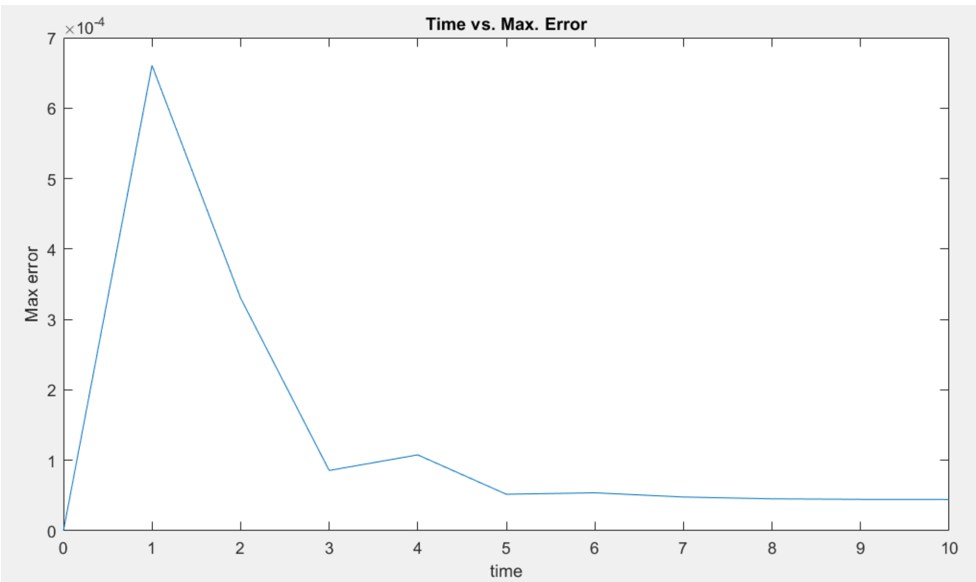

Figure 9: Max. Error vs. Time

It can be observed that as time advances, the change in the error of the numerical solution between consecutive time steps decreases, which can be explained by the fact that as time advances, the value of the exponential term e^{-3t} starts to approximate zero. Therefore, for a large enough final time, the effect of time in the solution will be negligible.

2.3 Part 3

As discussed in *Part 2*, for $t_{end} = 50$, the time depending term $e^{-3t} \approx 0$. Hence, the solution no longer depends on time, the problem becomes **stationary**, and the PDE takes the **eliptic** form:

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad\qquad in \quad \Omega = [0,1]^2 \tag{11}$$

And the Dirichlet boundary condition:

$$u = x^2 \qquad\qquad for \quad y = 0 \tag{12}$$

Making the corresponding changes in the MATLAB PDE Toolbox, using the same meshes and following the same procedure developed in *Part 1*, we obtain the following results:

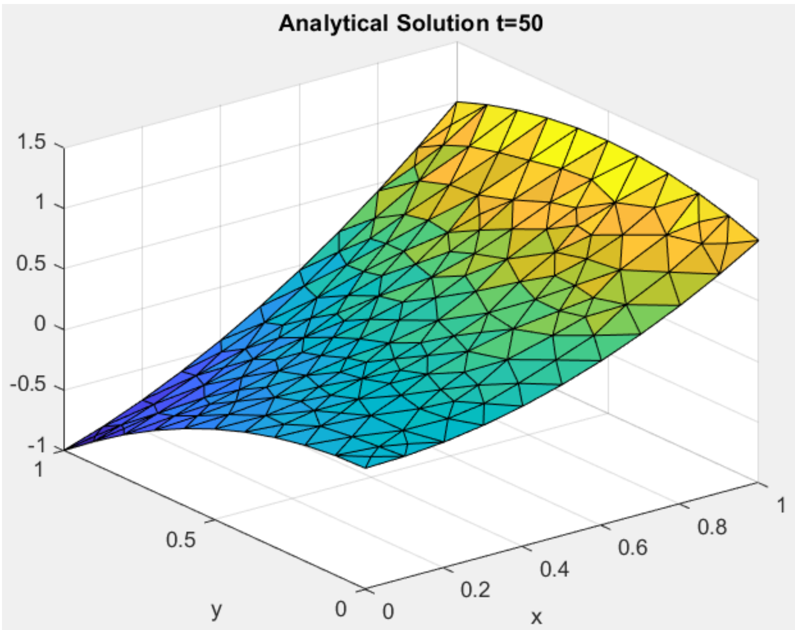

Figure 10: Analytical Solution, t=50

Mesh	Max. Error	Elements	h
Initial	0.0067	328	0.0781
Refined 1	0.0020	1312	0.0390
Refined 2	5.6788e-04	5248	0.0195
Refined 3	1.5992e-04	20992	9.76e-03
Refined 4	4.4414e-05	83968	4.88e-03

Table 3: Max. Error vs. Element Size, simplified problem for t=50

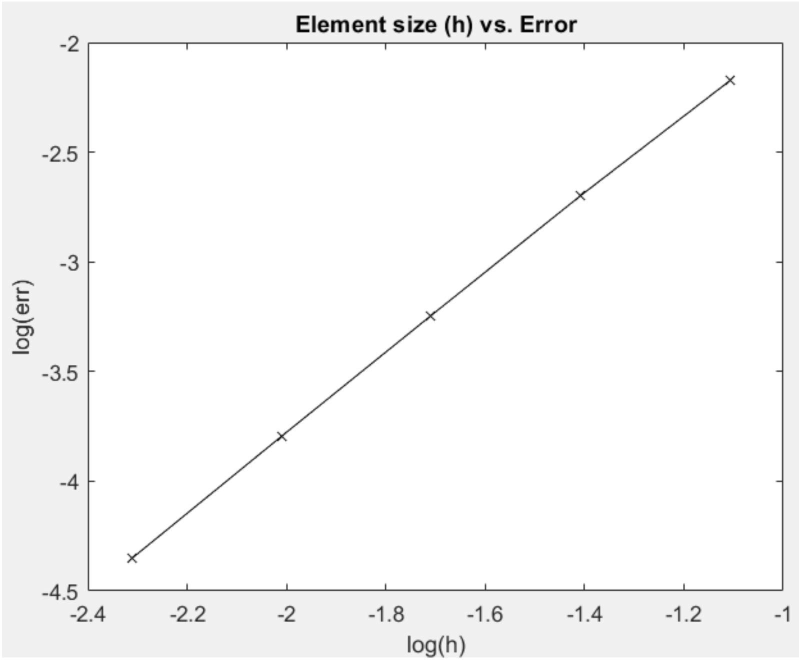

Figure 11: Max Error vs. Element Size, simplified problem for t=50

Since the meshes are the same as the ones employed in *Part 1*, the element sizes h did not change. The errors between the analytical solution and the numerical solution for the simplified problem associated to each h did not show a significant variation when compared to those found in *Part 1*. Therefore, we can conclude that eliminating time dependence from the solution at $t = 50$ does not affect its precision in any significant manner.

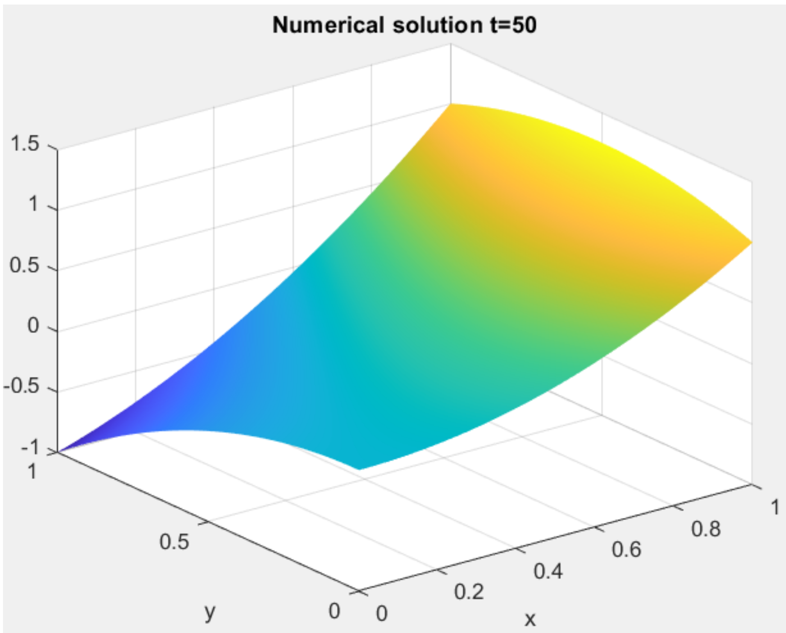

Figure 12: Numerical solution for refined mesh, simplified problem for t=50

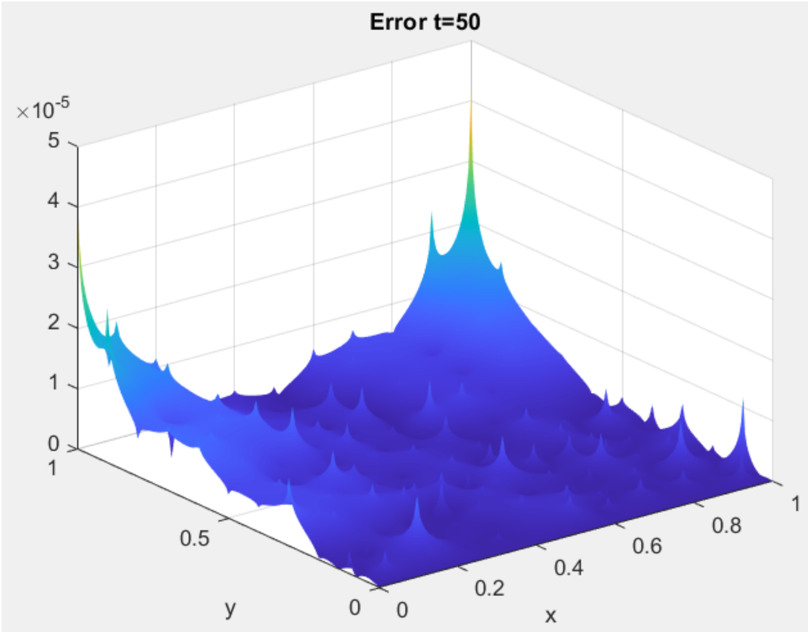

Figure 13: Error in numerical solution for refined mesh, simplified problem for t=50

From the precision in the numerical solution found for the simplified problem with the refined mesh (83968 elements) we conclude that the assumptions made are correct and the problem can be considered as stationary.