
Computational Mechanics Tools

Homework 3 – PDE ToolBox

Submitted By : Kiran Kolhe

Submitted To : Prof. Amir Abdollahi

12th December 2019

Contents

✚ Details	1
✚ Task 1.....	2
✚ Task 2.....	4
✚ Task 3.....	5

Details :

In this assignment of MATLAB PDE Toolbox, we have been asked to solve a PDE and observe its evolution over different time intervals checking for the convergence of the solution.

The given data is:

Problem equation: $u_t - \Delta u = f$ in $\Omega [0, 1]^2$

Source Term: $f(x, y, t) = -3e^{-3t}$

Initial condition: At $t=0$, $u(x, y, t) = x^2 + xy - y^2 + 1$

Boundary Conditions:

$$\begin{aligned}u_n(x = 0, y, t) &= -y \\u_n(x = 1, y, t) &= 2 + y \\u_n(x, y = 0, t) &= x^2 + e^{-3t} \\u_n(x, y = 1, t) &= x - 2\end{aligned}$$

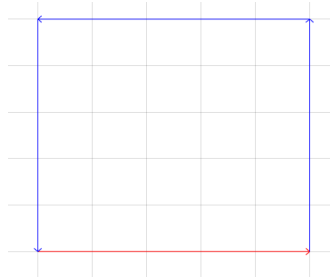
Analytical Solution: $u(x, y, t) = x^2 + xy - y^2 + e^{-3t}$

❖ Tasks to be done:

1. Consider $t_{end} = 10$, solve the problem, and refine the initial mesh up to 4 times. Verify that the theoretical convergence order holds.
2. How is the solution affected when we modify the final time?
3. We are interested in obtaining the solution at time $t_{end} = 50$. Find a more efficient manner to solve this problem. You do not need to prove the equivalence mathematically, but you need to provide numerical evidence of the new method.

Task 1 :

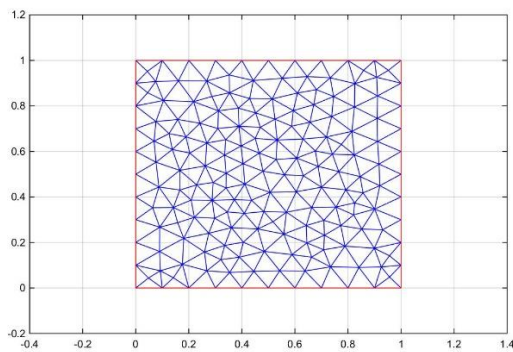
Observing the given PDE, it is seen that the PDE is of parabolic type. First, we define the geometry of the polygon in the PDE Toolbox. The co-ordinates of the polygon are (0,0) (1,0) (1,1) and (0,1). Then the boundary conditions are imposed on the geometry. The lower boundary (red) is a Dirichlet boundary while the rest all boundaries (blue) deal with the Neumann boundary.



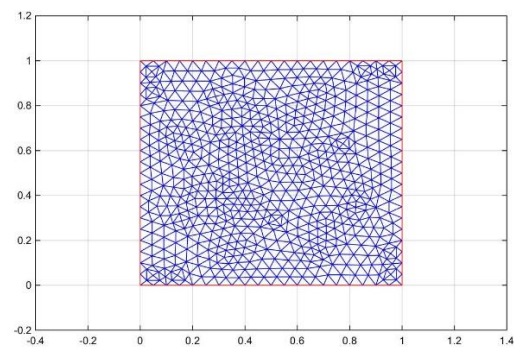
Geometry defined by Boundary Conditions

After setting the boundaries, parabolic PDE type is selected and the parameters are set to the respective values, $a = 0$, $d = 1$, $c = 1$. The time interval ($t_{end} = 10$) is set upto 10 sec and initial condition is given to the PDE. After this, the PDE is solved and then mesh is refined 4 times and solved again. In each refinement, the error is calculated between the analytical solution and the numerical solution.

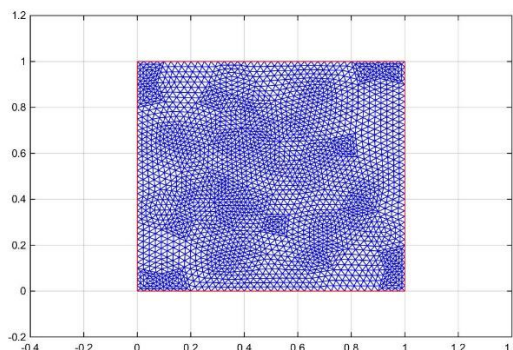
The initial mesh and 4 mesh refinements are indicated below:



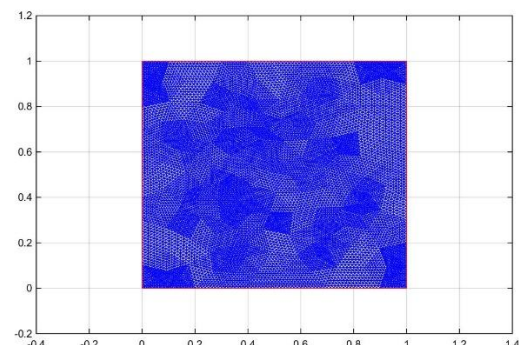
Initial Mesh



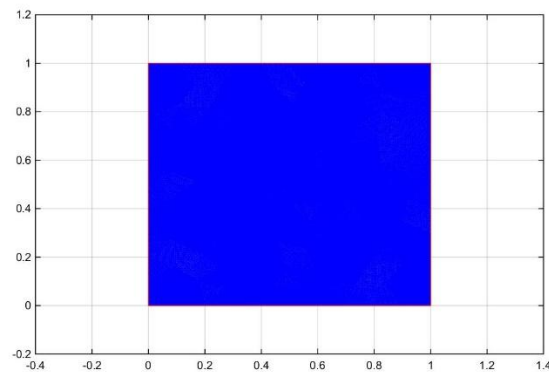
1st refinement



2nd refinement



3rd refinement

4th refinement

Here, we have 4 mesh refinements from the initial mesh, hence 5 data points. Listed below is the table of element size (h) and max. error between analytical and numerical solution for that respective mesh refinement.

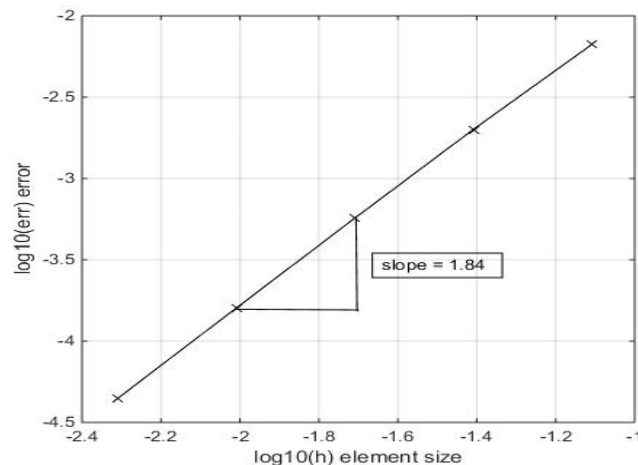
	Element size (h)	Max. Error (err)
Initial Mesh	0.0781	0.0067
Refinement 1	0.0390	0.0020
Refinement 2	0.0195	5.6787e-04
Refinement 3	0.0098	1.6004e-04
Refinement 4	0.0049	4.4423e-05

The element size ' h ' is calculated by $A = N * \frac{h^2}{2}$

where A is the area of the domain and N is the number of elements in the mesh.

This ' h ' can be calculated as we know the area, which is $A = 1$.

For the convergence observation, a logarithmic plot of Element Size against Max. Error is plotted and it is observed that the slope of the plot is approximately 1.84 which is close to the theoretical value of 2 for this problem. Thus, the convergence holds true.

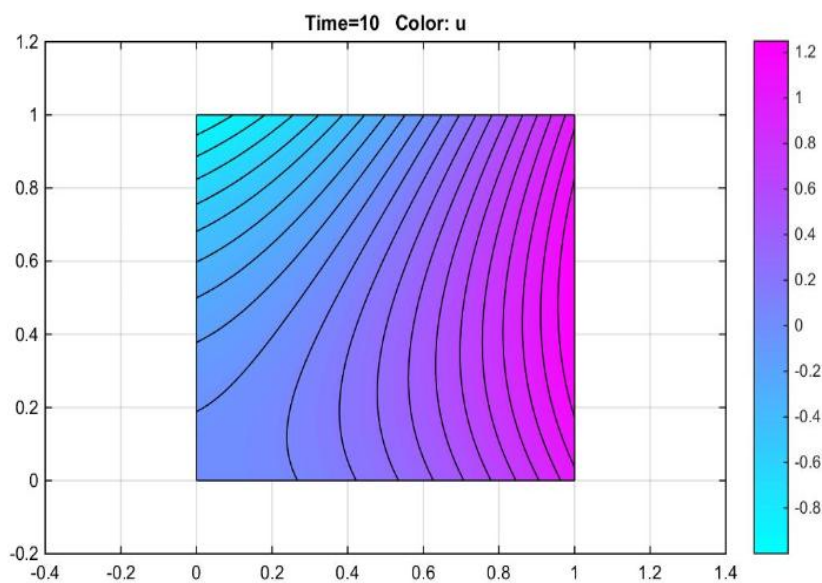


Plot of element size (h) against Max. error in each mesh refinement for $t_{\text{end}} = 10$

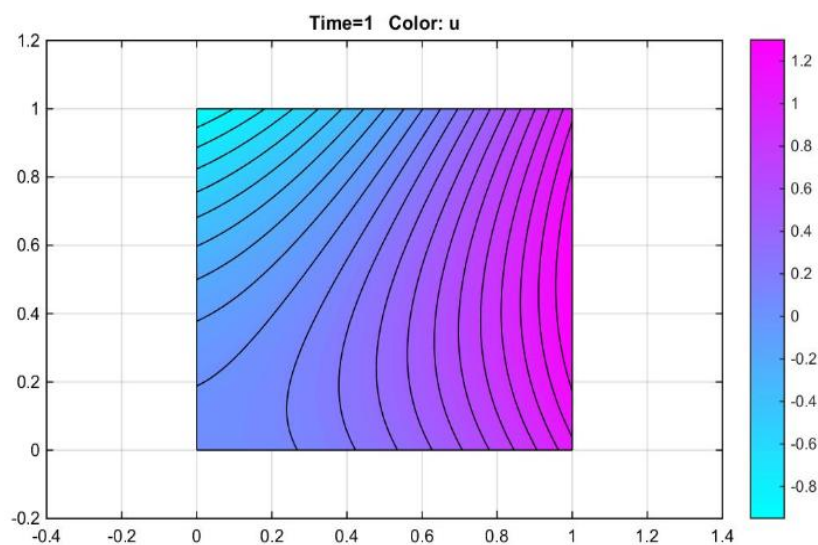
Task 2 :

The effect of the final time on the solution is found by comparing the results obtained for $t_{end} = 10$ and $t_{end} = 1$. The same problem is checked for $t_{end} = 1$ and the Max. error is observed. It is observed from the solution, that changing the final time t_{end} has negligible effects on the results obtained. This can be explained by the fact that, the value of source term 'f' approaches a very small value as time increases, and as a matter of fact tends to zero because the contribution from this term to the final value is negligible after some time. Therefore, it can be regarded as a steady state problem after some seconds.

The results of the problem for $t_{end} = 10$ and $t_{end} = 1$ are shown below.



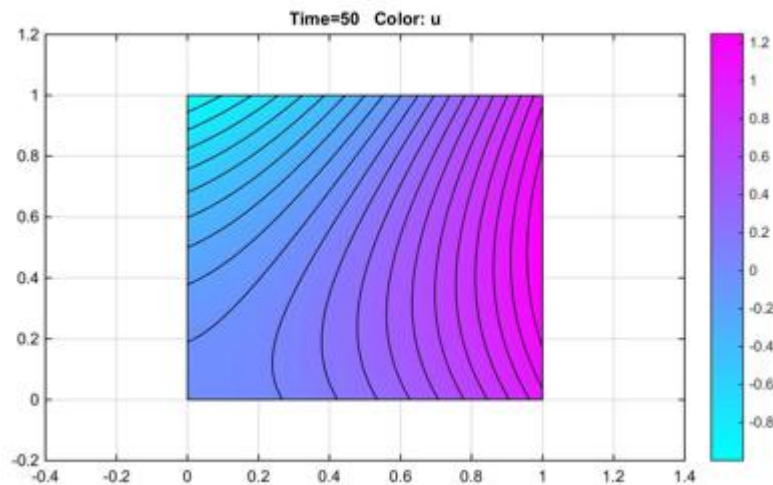
Result for $t_{end} = 10$



Result for $t_{end} = 1$

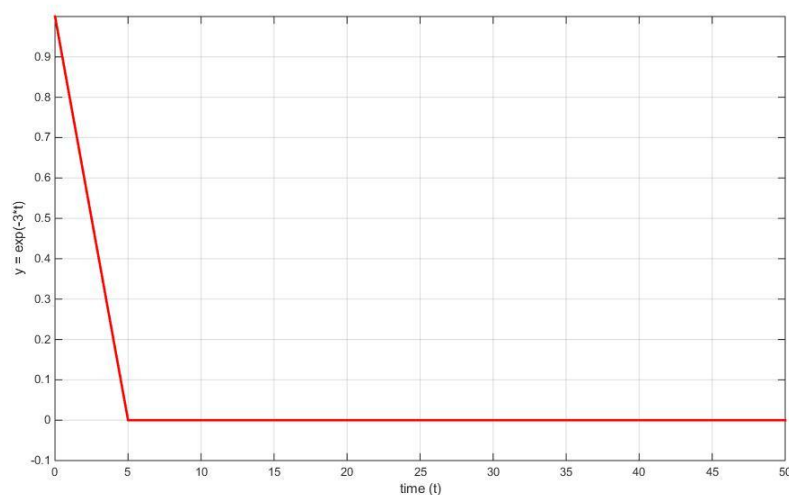
Task 3 :

The result for $t_{end} = 50$ is shown below.



Result for $t_{end} = 50$

It is observed that there is not much difference in the evolution of the problem for $t_{end} = 50$. As explained earlier, increasing the time factor does not have much effect on the solution after a particular time, because the e^{-3t} term tends to have a very less value with increasing time. The evolution of the term e^{-3t} can be observed graphically as well. It seems that this term underdoes a negligible constant value after some time which does not change further with time.



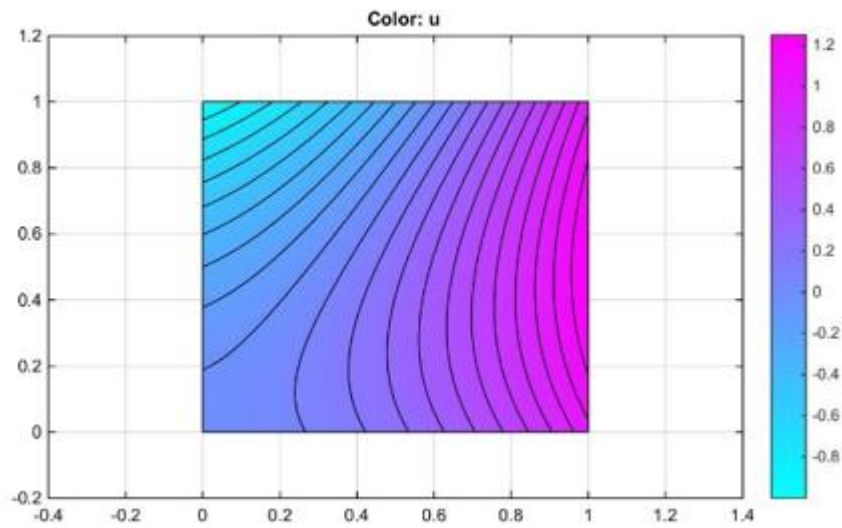
Plot of e^{-3t} against time

So, in particular, as the term e^{-3t} has negligible values, it can be neglected from the equations in which it is present. The analytical equation, source term and the Dirichlet Boundary all of them contain this exponential term. So, if we ignore e^{-3t} from all of them, the problem becomes time independent.

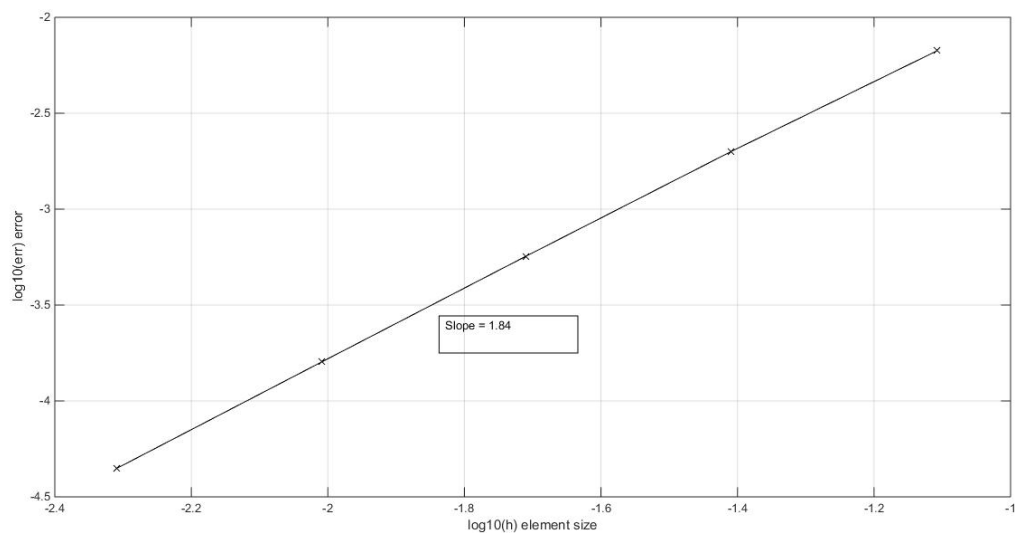
Thus, the main PDE becomes $\Delta u = f$.

It can be observed that making the PDE time independent changes it to elliptical PDE. The problem is now a steady state problem. Now, if we compute this PDE and observe the results, they don't show much difference from the previous ones. It can thus be concluded

that solving the given PDE as a steady state problem is thus more efficient technique in computational point of view.



Result of other technique of solving the problem



Plot of element size (h) against Max. error in each mesh refinement for new method

It is also observed that the slope of the solution by new method is almost equal to the slope of the solution by old method.