Coupling in Time II

Fractional Step Schemes

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

- Let us consider the incompressible Navier-Stokes equations :

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \mathbf{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with boundary conditions:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \qquad \text{on } \Gamma_D$$
$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{t} \qquad \text{on } \Gamma_N$$

and initial conditions:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0$$

- Pressure is determined up to an arbitrary additive constant. It is the Lagrange multiplier for the incompressibility constraint.

- The problem when solving these equations is that **pressure and velocity are coupled**.

- They cannot be solved independently, thus **the computational cost is high.**

- Stabilization methods are required, due to both the convective term and the LBB inf-sup condition which velocity-pressure interpolation spaces must satisfy

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

The objective is to decouple velocity and pressure in order to reduce computational cost.

Classical fractional step methods consist of splitting these equations in two steps.

Suppose that we are using a backward Euler time integration scheme. The first step is: find an intermediate velocity $\widehat{\boldsymbol{u}}^{n+1}$ such that:

$$\frac{\widehat{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} - \nu \Delta \widehat{\boldsymbol{u}}^{n+1} + \widehat{\boldsymbol{u}}^{n+1} \cdot \nabla \widehat{\boldsymbol{u}}^{n+1} = \boldsymbol{f}^{n+1}$$

$$\widehat{\boldsymbol{u}}^{n+1} = \overline{\boldsymbol{u}} \qquad \text{on } \Gamma_D$$

- Note that we have eliminated the pressure unknown and the continuity equation
- This is simply an **advection-diffusion equation**. Plus, if usual boundary conditions are imposed, **velocity unknowns are decoupled**.
- The intermediate velocity is not satisfying the incompressibility constraint

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

The second step consists in finding a end of step velocity $\boldsymbol{u}^{n+1} \in \boldsymbol{J}_0$ and a pressure $p \in H^1$ such that:

$$\frac{\boldsymbol{u}^{n+1} - \widehat{\boldsymbol{u}}^{n+1}}{\delta t} + \nabla p^{n+1} = 0,$$
$$\nabla \cdot \boldsymbol{u}^{n+1} = 0,$$
$$\boldsymbol{n} \cdot \boldsymbol{u}^{n+1} = \boldsymbol{n} \cdot \overline{\boldsymbol{u}} \qquad \text{on } \Gamma_D$$

We note that this is equivalent to projecting $\widehat{\boldsymbol{u}}^{n+1}$ onto $\boldsymbol{J}_0$ the space of weakly divergence free functions:

$$\boldsymbol{J}_0 := \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot \boldsymbol{u} = 0\}$$

Note that, due to the Helmholtz decomposition, for a sufficiently smooth $\widehat{\boldsymbol{u}}^{n+1}$, we can decompose $\widehat{\boldsymbol{u}}^{n+1}$ into:

$$\widehat{\boldsymbol{u}}^{n+1} = \widehat{\boldsymbol{u}}^{n+1}_{\nabla\times} + \widehat{\boldsymbol{u}}^{n+1}_{\nabla}$$

where:

$$\nabla \cdot \widehat{\boldsymbol{u}}^{n+1}_{\nabla\times} = 0$$
$$\widehat{\boldsymbol{u}}^{n+1}_{\nabla} = \nabla p \cdot \delta t$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

An error related to the pressure boundary conditions has been introduced!

If we take into account that:

$$\hat{\boldsymbol{u}}^{n+1} = \bar{\boldsymbol{u}} \qquad \text{on } \Gamma_D$$
$$\boldsymbol{n} \cdot \boldsymbol{u}^{n+1} = \boldsymbol{n} \cdot \bar{\boldsymbol{u}} \qquad \text{on } \Gamma_D$$

Then, due to:

$$\frac{\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^{n+1}}{\delta t} + \nabla p^{n+1} = 0,$$

the pressure field is satisfying an artificial boundary condition:

$$\boldsymbol{n} \cdot \nabla p^{n+1}\Big|_{\Gamma} = 0.$$

This introduces an spurious pressure boundary layer of width $\mathcal{O}\sqrt{\nu \Delta t}$

On the other hand, the method is stable for equal order interpolations if $\Delta t > Ch^2$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

However, velocity and pressure still have not been decoupled.

In order to uncouple them, we take the divergence of :

$$\frac{\boldsymbol{u}^{n+1} - \widehat{\boldsymbol{u}}^{n+1}}{\delta t} + \nabla p^{n+1} = 0,$$

which leads us to:

$$\Delta p^{n+1} = \frac{\nabla \cdot \widehat{\boldsymbol{u}}^{n+1}}{\delta t},$$
$$\boldsymbol{n} \cdot \nabla p^{n+1}\Big|_{\Gamma} = 0.$$

This is called the **Pressure Poisson Equation**.

Once the pressure is calculated, we recover the end of step velocity from the first equation.

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Improving the splitting error

Let us consider the fractional step equations:

$$\frac{\hat{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^n}{\delta t}-\nu\Delta\hat{\boldsymbol{u}}^{n+1}+\hat{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{n+1}=\boldsymbol{f}^{n+1}$$

$$\frac{\boldsymbol{u}^{n+1}-\hat{\boldsymbol{u}}^{n+1}}{\delta t}+\nabla p^{n+1}=0,$$

$$\nabla\cdot\boldsymbol{u}^{n+1}=0,$$

From the second equation we can see that:

$$\mathcal{O}\left|\left|\boldsymbol{u}^{n+1}-\hat{\boldsymbol{u}}^{n+1}\right|\right|\approx\delta t\,\mathcal{O}\left|\left|p^{n+1}\right|\right|$$

An improvement in the splitting error:

$$\frac{\hat{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^n}{\delta t}-\nu\Delta\hat{\boldsymbol{u}}^{n+1}+\hat{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{n+1}+\boldsymbol{\nabla}\boldsymbol{p}^{\boldsymbol{n}}=\boldsymbol{f}^{n+1}$$

$$\frac{\boldsymbol{u}^{n+1}-\hat{\boldsymbol{u}}^{n+1}}{\delta t}+\left(\nabla p^{n+1}-\boldsymbol{\nabla}\boldsymbol{p}^{\boldsymbol{n}}\right)=0,$$

$$\nabla\cdot\boldsymbol{u}^{n+1}=0,$$

Now the splitting error is:

$$O\left|\left|\boldsymbol{u}^{n+1}-\hat{\boldsymbol{u}}^{n+1}\right|\right|\approx\delta t\,O\left|\left|p^{n+1}-p^n\right|\right|\approx\boldsymbol{\delta t^2}\,\boldsymbol{\mathcal{O}}\left|\left|\boldsymbol{p}^{\boldsymbol{n+1}}\right|\right|$$

Stability for equal-order interpolations is also enhanced, stable for $\delta t > Ch$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Predictor-corrector schemes

Departing from the 2nd order fractional step method, we define an iterative procedure to converge to the monolithic scheme:

While convergence is not reached, iterate:

Step 1:

$$\frac{\hat{\boldsymbol{u}}^{n+1,i+1}-\boldsymbol{u}^{n}}{\delta t}-\nu\Delta\hat{\boldsymbol{u}}^{n+1,i+1}+\boldsymbol{u}^{n+1,i}\cdot\nabla\hat{\boldsymbol{u}}^{n+1,i+1}+\nabla p^{n+1,i}=\boldsymbol{f}^{n+1}$$

Step 2:

$$\Delta p^{n+1,i+1}=\frac{\nabla\cdot\hat{\boldsymbol{u}}^{n+1,i+1}}{\delta t},$$

Step 3:

$$\frac{\boldsymbol{u}^{n+1,i+1}-\hat{\boldsymbol{u}}^{n+1,i+1}}{\delta t}+\nabla p^{n+1,i+1}=0$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Momentum-pressure Poisson equation methods

Instead of solving the continuity equation, we replace it by a Pressure Poisson Equation:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\mathbf{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f},$$
$$\Delta p = \nabla \cdot (\boldsymbol{f} + \nu\Delta\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\mathbf{u}).$$

Note that the second equation is obtained by taking the divergence of the first one and taking into account that:

$$\nabla \cdot \partial_t \boldsymbol{u} = \partial_t \nabla \boldsymbol{u} = 0.$$

After discretizing in time we can write:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} - \nu\Delta\boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \boldsymbol{\nabla}\boldsymbol{u}^{n+1} + \nabla p^* = \boldsymbol{f}^{n+1},$$
$$\Delta p^{n+1} = \nabla \cdot (\boldsymbol{f}^{n+1} + \nu\Delta\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1} \cdot \boldsymbol{\nabla}\boldsymbol{u}^{n+1}).$$

Where p^* is a high order extrapolation of p^{n+1}. For instance:

$$p^* = 2p^n - p^{n-1}$$

But the incompressibility constraint is relaxed!

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Velocity correction schemes

We can improve the fulfillment of the incompressibility constraints by extrapolating the velocity instead of the pressure (and solving first for the PPE):

$$\Delta p^{n+1} = \nabla \cdot (\boldsymbol{f}^{n+1} + \nu \Delta \boldsymbol{u}^{*} - \boldsymbol{u}^{*} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{*}).$$

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}}{\delta t} - \nu \Delta \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{n+1} + \nabla p^{n+1} = \boldsymbol{f}^{n+1},$$

The problem of this type of approximation is that it requires a higher regularity for the interpolation space of $\boldsymbol{u}$: we require $\boldsymbol{u} \in H^2$.

This is so because the finite element weak form of the PPE reads:

$$(\nabla q_h, \nabla p_h^{n+1}) = (\nabla q_h, \boldsymbol{f}^{n+1} + \nu \Delta \boldsymbol{u}_h^{*} - \boldsymbol{u}_h^{*} \cdot \boldsymbol{\nabla} \boldsymbol{u}_h^{*}).$$

This issue can be circumvented by adopting an algebraic approach

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Algebraic approach

Let us consider the discrete weak form of the incompressible Navier-Stokes equations (stabilization is required!):

$$\left(\boldsymbol{v}_h, \frac{\boldsymbol{u}_h^{n+1}}{\delta_t}\right) + \left(\boldsymbol{v}_h, \boldsymbol{u}_h^{n+1} \cdot \nabla \boldsymbol{u}_h^{n+1}\right) + \nu\left(\nabla v_h, \nabla \boldsymbol{u}_h^{n+1}\right) - \left(\nabla \cdot \boldsymbol{v}_h, p_h^{n+1}\right)$$

$$= (\boldsymbol{v}_h, \boldsymbol{f}) + \left(\boldsymbol{v}_h, \frac{\boldsymbol{u}^n}{\delta_t}\right),$$

$$\left(q_h, \nabla \cdot \boldsymbol{u}_h^{n+1}\right) = 0.$$

We can write it in an algebraic form:

$$\boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n+1} + \boldsymbol{K}(\boldsymbol{U}^{n+1})\boldsymbol{U}^{n+1} + \boldsymbol{G}\boldsymbol{P}^{n+1} = \boldsymbol{F}^{n+1} - \boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n}$$

$$\boldsymbol{D}\boldsymbol{U}^{n+1} = 0,$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Algebraic approach

We multiply the first equation by $\boldsymbol{DM}^{-1}$, and we replace it by the continuity equation:

$$\boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n+1} + \boldsymbol{K}(\boldsymbol{U}^{n+1})\boldsymbol{U}^{n+1} + \boldsymbol{G}\boldsymbol{P}^{n+1} = \boldsymbol{F}^{n+1} - \boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n}$$
$$\boldsymbol{DM}^{-1}\boldsymbol{G}\boldsymbol{P}^{n+1} = \boldsymbol{DM}^{-1}\boldsymbol{F}^{n+1} - \boldsymbol{DM}^{-1}\boldsymbol{K}\left(\boldsymbol{U}^{n+1}\right)\boldsymbol{U}^{n+1} + \boldsymbol{DU}^{n},$$

This is exactly equivalent to the monolithic formulation, and there are no regularity requirements.

Plus, $\boldsymbol{DM}^{-1}\boldsymbol{G}$ is a discrete version of the Laplacian operator. We can replace it by the laplacian operator

$$\boldsymbol{DM}^{-1}\boldsymbol{G} \approx \boldsymbol{L}$$

but then we introduce the boundary conditions issue. A better approximation is:

$$\boldsymbol{DM}^{-1}\boldsymbol{G}\boldsymbol{P}^{n+1} = \boldsymbol{L}\boldsymbol{P}^{n+1} + (\boldsymbol{DM}^{-1}\boldsymbol{G} - \boldsymbol{L})\boldsymbol{P}^{n+1} \approx \boldsymbol{L}\boldsymbol{P}^{n+1} + \left(\boldsymbol{DM}^{-1}\boldsymbol{G} - \boldsymbol{L}\right)\boldsymbol{P}^{*}$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Velocity projection methods

Using the previous approximation for $\boldsymbol{DM^{-1}G}$, and introducing an approximation for the velocity at $n+1$ we obtain the following velocity correction method:

Step 1. Solve for the pressure:

$$\boldsymbol{L}\boldsymbol{P}^{n+1} = \boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{F}^{n+1} - \boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{K}(\boldsymbol{U}^{*})\boldsymbol{U}^{*} + \boldsymbol{D}\boldsymbol{U}^{n} - \left(\boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{G} - \boldsymbol{L}\right)\boldsymbol{P}^{*}$$

Step 2. Solve for the velocity:

$$\boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n+1} + \boldsymbol{K}(\boldsymbol{U}^{n+1})\boldsymbol{U}^{n+1} = \boldsymbol{F}^{n+1} - \boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n} - \boldsymbol{G}\boldsymbol{P}^{n+1}$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Predictor corrector schemes

We can define an iterative procedure on the previous velocity correction scheme:

While not converged, iterate:
Step 1. Solve for the pressure:

$$\boldsymbol{L}\boldsymbol{P}^{n+1,i+1} = \boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{F}^{n+1} - \boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{K}\left(\boldsymbol{U}^{n+1,i}\right)\boldsymbol{U}^{n+1,i} + \boldsymbol{D}\boldsymbol{U}^{n} - \left(\boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{G} - \boldsymbol{L}\right)\boldsymbol{P}^{n+1,i}$$

Step 2. Solve for the velocity:

$$\boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n+1,i+1} + \boldsymbol{K}\left(\boldsymbol{U}^{n+1,i+1}\right)\boldsymbol{U}^{n+1,i+1} = \boldsymbol{F}^{n+1} - \boldsymbol{M}\frac{1}{\delta t}\boldsymbol{U}^{n} - \boldsymbol{G}\boldsymbol{P}^{n+1,i+1}$$

FRACTIONAL STEP SCHEMES FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Solving non-linear systems of equations

Let us suppose that we have a non-linear system of the type:

$$\boldsymbol{K}(\boldsymbol{U})\boldsymbol{U} = \boldsymbol{F}$$

where $\boldsymbol{K}(\boldsymbol{U})$ is linear on $\boldsymbol{U}$. The LHS is non-linear, it cannot be solved straight-forwardly.

Picard iterative scheme (fixed point iteration):

$$\boldsymbol{K}(\boldsymbol{U}^i)\boldsymbol{U}^{i+1} = \boldsymbol{F}$$

Newton iterative scheme:

$$\boldsymbol{U}^{i+1} = \boldsymbol{U}^i + \Delta\boldsymbol{U}^{i+1}$$

$$\boldsymbol{K}(\boldsymbol{U}^i + \Delta\boldsymbol{U}^{i+1})(\boldsymbol{U}^i + \Delta\boldsymbol{U}^{i+1}) = \boldsymbol{F}$$

$$\boldsymbol{K}(\boldsymbol{U}^i)(\boldsymbol{U}^i) + \boldsymbol{K}(\boldsymbol{U}^i + \Delta\boldsymbol{U}^{i+1})(\boldsymbol{U}^i) + \boldsymbol{K}(\boldsymbol{U}^i)(\boldsymbol{U}^i + \Delta\boldsymbol{U}^{i+1}) + \boldsymbol{K}(\Delta\boldsymbol{U}^{i+1})(\Delta\boldsymbol{U}^{i+1}) = \boldsymbol{F}$$

The final system is:

$$\boldsymbol{K}(\boldsymbol{U}^{i+1})(\boldsymbol{U}^i) + \boldsymbol{K}(\boldsymbol{U}^i)(\boldsymbol{U}^{i+1}) = \boldsymbol{F} - \boldsymbol{K}(\boldsymbol{U}^i)(\boldsymbol{U}^i)$$

Relaxation and under-relaxation are also possible: $\boldsymbol{U}^{i+1} = \boldsymbol{U}^i + w\Delta\boldsymbol{U}^{i+1} \quad w \in (0, \infty)$

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Operator splitting techniques

The idea is to decompose the system of PDEs into simpler problems. This allows to treat each problem with specific algorithms.

The splitting can be applied at the continuous (Differential Splitting) or at the discretized (Algebraic Splitting) levels.

- Operator splitting allows to treat some each of the split operators by using different temporal integrators. Particularly, explicit or implicit methods can be used.
- We can use different time steps for different problems. This option is particularly interesting if operators have very different time step restrictions for explicit schemes.
- Discretization in space can also be different for the various subproblems (finite elements, volumes, differences).
- Splitting errors depend on the time step size linearly, quadratically…

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Operator splitting techniques

Let us consider the convection-diffusion equation:

$$\partial_t u + \boldsymbol{a} \cdot \nabla u - \nu \Delta u = f \qquad \text{in } (0, \mathrm{T}) \qquad u_0 = 0$$

We define

$$\mathcal{L} = \mathcal{L}_{\boldsymbol{a}} + \mathcal{L}_{\nu}$$
$$\mathcal{L}_{\boldsymbol{a}} u = \boldsymbol{a} \cdot \nabla u$$
$$\mathcal{L}_{\nu} u = -\nu \Delta u$$

This yields:

$$\partial_t u + \mathcal{L}_{\boldsymbol{a}} u + \mathcal{L}_{\nu} u = f$$

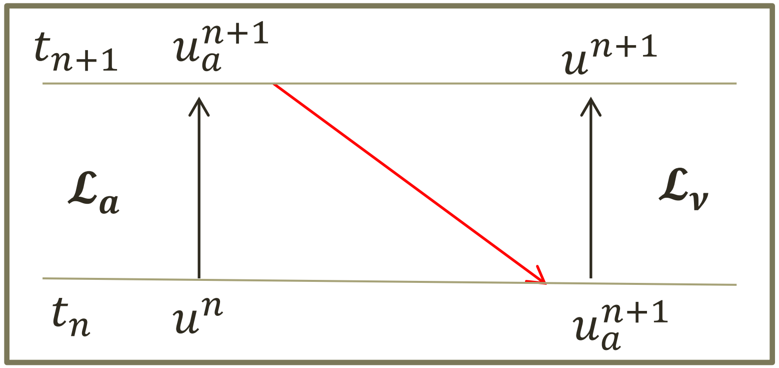
We introduce the splitting by defining intermediate variables u_a and u_ν.

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Operator splitting techniques

In order to advance in time, we first solve for u_a:

$$u_a(t_n) = u^n$$
$$\partial_t u_a + \boldsymbol{\mathcal{L}}_{\boldsymbol{a}} u_a = 0$$



Secondly we solve for u_v,
but the initial condition is $u_a(t_{n+1})$:

$$u_v(t_n) = u_a(t_{n+1})$$
$$\partial_t u_v + \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} u_v = f$$

Finally:

$$u^{n+1} = u_v(t_{n+1})$$

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Operator splitting techniques

$$\partial_t u + \boldsymbol{\mathcal{L}_a} u - \boldsymbol{\mathcal{L}_\nu} u = f$$

Let us discretize both problems by using a backward Euler scheme:

Step 1:

$$\frac{u_a^{n+1} - u^n}{\delta t} + \boldsymbol{\mathcal{L}_a} u_a^{n+1} = 0 \qquad \Rightarrow \qquad u_a^{n+1} = (\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_a})^{-\mathbf{1}} u^n$$

Step 2:

$$\frac{u^{n+1} - u_a^{n+1}}{\delta t} + \boldsymbol{\mathcal{L}_\nu} u^{n+1} = \boldsymbol{f} \qquad \Rightarrow \qquad u^{n+1} = (\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_\nu})^{-\mathbf{1}} (u_a^{n+1} + \boldsymbol{f} \delta t)$$

As a consequence:

$$u^{n+1} = (\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_\nu})^{-\mathbf{1}} \left[(\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_a})^{-\mathbf{1}} u^n + \boldsymbol{f} \delta t \right]$$

$$u^n = (\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_a}) [(\boldsymbol{I} + \delta t \boldsymbol{\mathcal{L}_\nu}) u^{n+1} - \boldsymbol{f} \delta t]$$

$$\boxed{\frac{u^{n+1} - u^n}{\delta t} + (\boldsymbol{\mathcal{L}_a} + \boldsymbol{\mathcal{L}_\nu}) u^{n+1} = \boldsymbol{f} + \delta t \boldsymbol{\mathcal{L}_a} (\boldsymbol{f} - \boldsymbol{\mathcal{L}_\nu} u^{n+1})}$$

The splitting error is $\boldsymbol{\mathcal{O}}(\delta t)$

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Operator splitting techniques

Consider a convective-dominated flow, where $a > \nu$

The advantage of the operator splitting is apparent if we take into account:

- The Courant-Friedrichs-Lewy condition for the convection diffusion equation is:

$$\delta t \leq \left(c_\nu \frac{\nu}{h^2} + c_a \frac{|\boldsymbol{a}|}{h} \right)^{-1}$$

- If we have a fine mesh, the maximum time step size is going to be dominated by $\frac{h^2}{\nu}$.
- Solving (implicitly) a transient convection-dominated problem is much harder than solving the pure transient diffusion problem (for iterative multigrid solvers).

Then a reasonable solution is to use the operator splitting with:

$$\delta t \leq \frac{h}{c_a |\boldsymbol{a}|},$$

use an explicit method for the convection and an implicit method for the diffusion.

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Second order operator splitting

Step 1: Integrate from t_n to $t_{n+1/2}$:

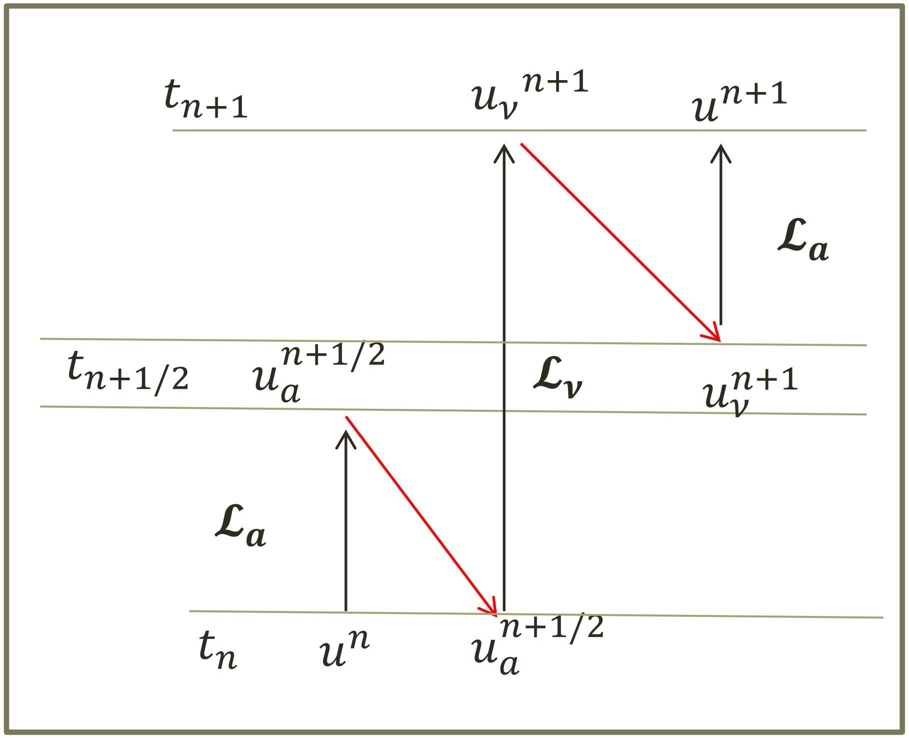
$$u_a^n = u^n$$
$$\partial_t u_a + \boldsymbol{\mathcal{L}}_{\boldsymbol{a}} u_a = 0$$

Step 2: Integragte from t_n to t_{n+1}, using $u_a^{n+1/2}$ as initial conditions:

$$u_v(t_n) = u_a(t_{n+1/2})$$
$$\partial_t u_v + \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} u_v = f$$

Step 3: Integrate from $t_{n+1/2}$ to t_{n+1}, using u_v^{n+1} as initial conditions :

$$u_a^n = u_v(t_{n+1})$$
$$\partial_t u_a + \boldsymbol{\mathcal{L}}_{\boldsymbol{a}} u_a = 0$$



OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Second order operator splitting

We consider a Crank-Nicolson discretization scheme (needs to be second order!)

Step 1:

$$2\frac{u_a^{n+1/2}-u^n}{\delta t}+\boldsymbol{\mathcal{L}_a}\left(\frac{u_a^{n+1/2}+u^n}{\mathbf{2}}\right)=0 \Rightarrow \quad u_a^{n+1/2}=\left(\boldsymbol{I}+\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)u^n$$

Step 2:

$$\frac{u_v^{n+1}-u_a^{n+1/2}}{\delta t}+\boldsymbol{\mathcal{L}_v}\left(\frac{u_v^{n+1}+u_a^{n+1/2}}{\mathbf{2}}\right)=0 \Rightarrow u_v^{n+1}=\left(\boldsymbol{I}+\frac{\delta t}{2}\boldsymbol{\mathcal{L}_v}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{2}\boldsymbol{\mathcal{L}_v}\right)u_a^{n+1/2}$$

Step 3:

$$2\frac{u^{n+1}-u_v^{n+1}}{\delta t}+\boldsymbol{\mathcal{L}_a}\left(\frac{u_a^{n+1}+u_v^{n+1}}{\mathbf{2}}\right)=0 \quad \Rightarrow \quad u^{n+1}=\left(\boldsymbol{I}+\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)u_v^{n+1}$$

As a consequence:

$$u^{n+1}=\left(\boldsymbol{I}+\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)\left(\boldsymbol{I}+\frac{\delta t}{2}\boldsymbol{\mathcal{L}_v}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{2}\boldsymbol{\mathcal{L}_v}\right)\left(\boldsymbol{I}+\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)^{\mathbf{-1}}\left(\boldsymbol{I}-\frac{\delta t}{4}\boldsymbol{\mathcal{L}_a}\right)u^n$$

The splitting error is $\boldsymbol{\mathcal{O}}(\delta t^2)$

OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Predictor corrector schemes

Predictor corrector schemes are also possible:

Step 1: Integrate from t_n to $t_{n+1/2}$:

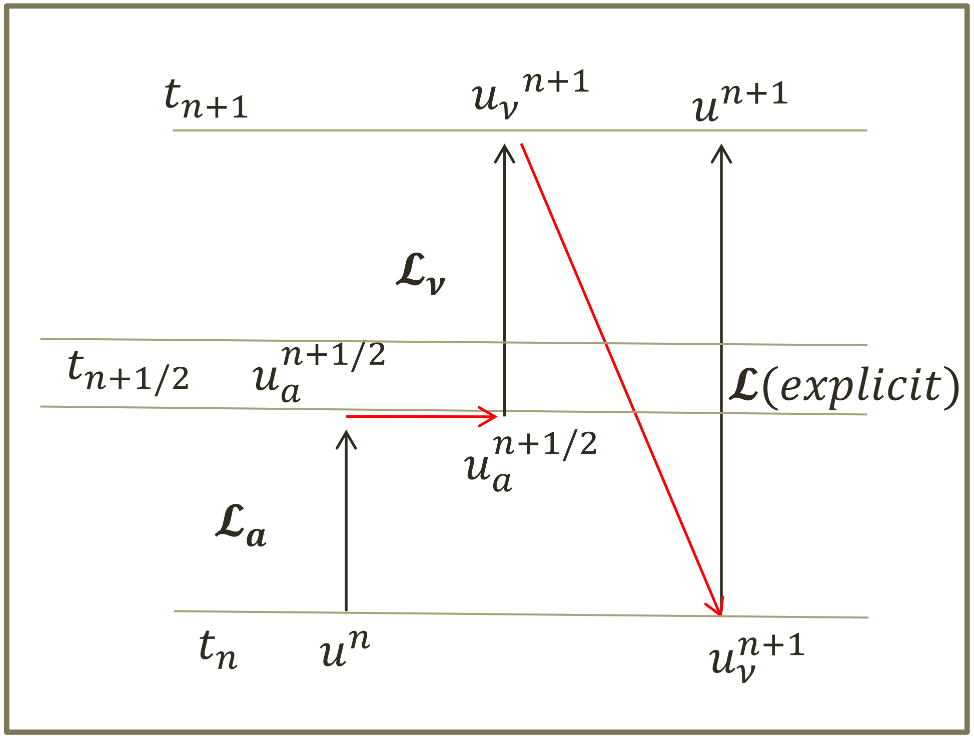
$$u_a^n = u^n$$
$$\partial_t u_a + \boldsymbol{\mathcal{L}}_{\boldsymbol{a}} u_a = 0$$

Step 2: Integrate from $t_{n+1/2}$ to t_{n+1}, using $u_a^{n+1/2}$ as initial conditions:

$$u_\nu(t_{n+1/2}) = u_a(t_{n+1/2})$$
$$\partial_t u_\nu + \boldsymbol{\mathcal{L}}_{\boldsymbol{\nu}} u_\nu = f$$

Step 3: Integrate from t_n to t_{n+1}, using u_ν^{n+1} as initial conditions (explicit) :

$$u^n = u_\nu(t_{n+1})$$
$$\partial_t u + \boldsymbol{\mathcal{L}} u = 0$$



OPERATOR SPLITTING FOR THE CONVECTION-DIFFUSION EQUATION

Predictor corrector schemes

Step 1:

$$2\frac{u_a^{n+1/2}-u^n}{\delta t}+\mathcal{L}_a u_a^{n+1/2}=\boldsymbol{f}^{n+1/2} \qquad \Rightarrow \qquad u_a^{n+1/2}=\left(\boldsymbol{I}+\frac{\delta t}{2}\mathcal{L}_a\right)^{-1}\left(u^n+\frac{\delta t}{2}\boldsymbol{f}^{n+1/2}\right)$$

Step 2:

$$2\frac{u_\nu^{n+1}-u_a^{n+1/2}}{\delta t}+\mathcal{L}_\nu u_\nu^{n+1}=0 \qquad \Rightarrow u_\nu^{n+1}=\left(\boldsymbol{I}+\frac{\delta t}{2}\mathcal{L}_\nu\right)^{-1}\left(u_a^{n+1/2}\right)$$

Step 3 (explicit integration with the full operator):

$$\frac{u^{n+1}-u_\nu^{n+1}}{\delta t}+\mathcal{L}u_\nu^{n+1}=\boldsymbol{f}^{n+1/2} \qquad \Rightarrow u^{n+1}=(\boldsymbol{I}+\delta t\mathcal{L})^{-1}\left(u_\nu^{n+1}+\delta t\boldsymbol{f}^{n+1/2}\right)$$

As a consequence (replacing $u_a^{n+1/2}$ in equation 2) :

$$\frac{u_\nu^{n+1}-u^n}{\delta t}+\frac{1}{2}\mathcal{L}u_\nu^{n+1}+\frac{\delta t}{4}\mathcal{L}_a\mathcal{L}_\nu u_\nu^{n+1}=\frac{1}{2}\boldsymbol{f}^{n+1/2}$$

Using equation 3 (instead of $\mathcal{L}u_\nu^{n+1}$):

$$u_\nu^{n+1}=\frac{u^{n+1}+u^n}{2}-\frac{\delta t^2}{4}\mathcal{L}_a\mathcal{L}_\nu u_\nu^{n+1}$$

Finally, replacing in equation 3:

$$\frac{u^{n+1}-u^n}{\delta t}+\mathcal{L}\left(\frac{u^{n+1}+u^n}{2}\right)=\boldsymbol{f}^{n+1/2}+\frac{\delta t^2}{4}\mathcal{L}\mathcal{L}_a\mathcal{L}_\nu u_\nu^{n+1}$$

Equivalent to Crank-Nicolson

OPERATOR SPLITTING AND FRACTIONAL STEP FOR INCOMPRESSIBLE NAVIER-STOKES

Step 1, convection:

$$\frac{\boldsymbol{u}_a^{n+1} - \boldsymbol{u}^n}{\delta t} + \boldsymbol{u}_a^{n+1} \cdot \nabla \boldsymbol{u}_a^{n+1} = \boldsymbol{0}$$

Step 2, diffusion:

$$\frac{\boldsymbol{u}_\nu^{n+1} - \boldsymbol{u}^n}{\delta t} - \nu \Delta \boldsymbol{u}_\nu^{n+1} = \boldsymbol{f}$$

Step 3, Pressure Poission equation:

$$\Delta p^{n+1} = \frac{\nabla \cdot \boldsymbol{u}_\nu^{n+1}}{\delta t}$$

Step 4: End of step velocity

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}_\nu^{n+1}}{\delta t} + \nabla p^{n+1} = 0,$$

Other combinations involving p^n and higher order approximations can be devised.

This could also be applied to velocity correction schemes