1. TRANSMISSON CONDITIONS IN CONTINUUM MECHANICS

1. TRANSMISSION CONDITIONS FOR POISSON'S PROBLEM.

Problem:

$$\left.\begin{aligned} -\nabla\cdot k\nabla u &= f \quad \text{in } \Omega \\ u &= \bar{u} \quad \text{on } \partial\Omega \end{aligned}\right\}$$

Variational formulation.

Let δu be such that $\delta u = 0$ on $\partial\Omega$.

$$\int_\Omega \delta u\,(-\nabla\cdot k\nabla u) = \int_\Omega \delta u\, f \qquad \forall\, \delta u$$

$$\int_\Omega \delta u\,(-\nabla\cdot k\nabla u) = \int_\Omega -\nabla\cdot(k\,\delta u\,\nabla u) + \int_\Omega k\,\nabla\delta u\cdot\nabla u$$

$$= \cancelto{0}{\int_{\partial\Omega} -\frac{\partial u}{\partial n}\, k\,\delta u} + \int_\Omega k\,\nabla\delta u\cdot\nabla u$$

$$\boxed{\int_\Omega k\,\nabla\delta u\cdot\nabla u = \int_\Omega \delta u\, f} \qquad \begin{aligned} &\forall\,\delta u \mid \delta u = 0 \text{ on } \partial\Omega \\ &u = \bar{u} \text{ on } \partial\Omega \end{aligned}$$

This is the so called weak form of the problem. Equivalently:

$$\boxed{u = \arg\inf_{\tilde{u}}\left[\frac{1}{2}\int_\Omega k|\nabla\tilde{u}|^2 - \int_\Omega \tilde{u}f\right]} \quad \text{Optimization problem.}$$

Let us prove this. Let

$$F(\tilde{u}) = \frac{1}{2}\int_\Omega k|\nabla\tilde{u}|^2 - \int_\Omega \tilde{u}f$$

F has a minimum at $u \Leftrightarrow \varphi(\varepsilon) := F(u+\varepsilon\,\delta u)$ has a minimum at $\varepsilon = 0 \Leftrightarrow \varphi'|_{\varepsilon=0} = 0$ & $\varphi''|_{\varepsilon=0} \geq 0$.

$$\left.\frac{d}{d\varepsilon}\varphi\right|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left[\frac{1}{2}\int_\Omega k(\nabla u+\varepsilon\nabla\delta u)\cdot(\nabla u+\varepsilon\nabla\delta u) - \int_\Omega (u+\varepsilon\,\delta u)f\right]\Bigg|_{\varepsilon=0}$$

$$= \left[\int_\Omega k(\nabla u+\varepsilon\nabla\delta u)\cdot\nabla\delta u - \int_\Omega \delta u\, f\right]\Bigg|_{\varepsilon=0}$$

$$= \int_\Omega k\,\nabla u\cdot\nabla\delta u - \int_\Omega \delta u f = 0 \Leftrightarrow \text{the weak form holds.}$$

$$\left.\frac{d^2}{d\varepsilon^2}\varphi\right|_{\varepsilon=0} = \int_\Omega k|\nabla\delta u|^2 \geq 0.$$ The critical point is a minimum.

Regularity

Let

$$L^2(\Omega) := \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < \infty \right\}$$

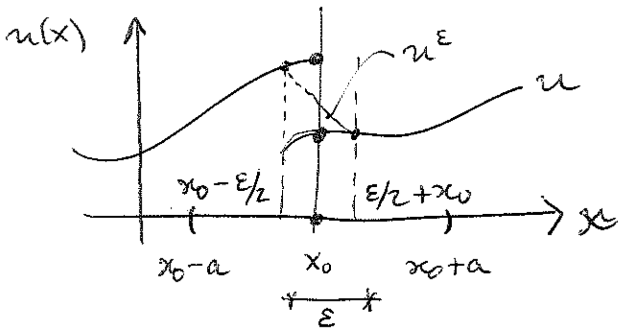
$$H^1(\Omega) := \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < \infty , \int_\Omega |\nabla v|^2 < \infty \right\}$$

Either the weak form or the optimization problem make sense if $u \in H^1(\Omega)$.

If only $u \in H^1(\Omega)$, the equation $-\nabla \cdot k \nabla u = f$ is said to hold in the sense of distributions.

Fact: if u is discontinuous across a surface (curve in 2D) it cannot be in $H^1(\Omega)$

Let us show this in 1D.



u^ε: regularized function

" $u = \lim_{\varepsilon \to 0} u^\varepsilon$ "

Let us assume that

$$\left.\frac{du}{dx}\right|_{x_0} = \lim_{\varepsilon \to 0} \left.\frac{d_\varepsilon u^\varepsilon}{dx}\right|_{x_0}$$

$$\int_{x_0-a}^{x_0+a} \frac{du^\varepsilon}{dx} = \int_{x_0-a}^{x_0-\varepsilon/2} \frac{du}{dx} + \int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} \frac{du^\varepsilon}{dx} + \int_{x_0+\varepsilon/2}^{x_0+a} \frac{du}{dx}$$

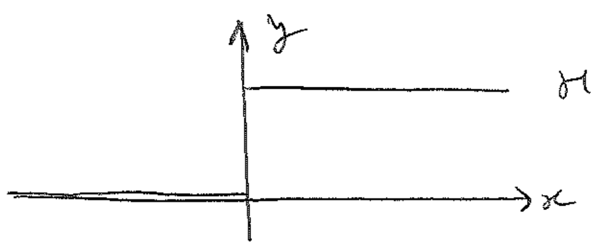
$$= \int_{x_0-a}^{x_0-\varepsilon/2} \frac{du}{dx} + \cancel{\varepsilon} \left[\frac{u(x_0+\varepsilon/2) - u(x_0-\varepsilon/2)}{\cancel{\varepsilon}} \right] + \int_{x_0+\varepsilon/2}^{x_0+a} \frac{du}{dx}$$

$$\xrightarrow{\varepsilon \to 0} \int_{x_0-a}^{x_0} \frac{du}{dx} + \left[u(x_0^+) - u(x_0^-) \right] + \int_{x_0}^{x_0+a} \frac{du}{dx} .$$

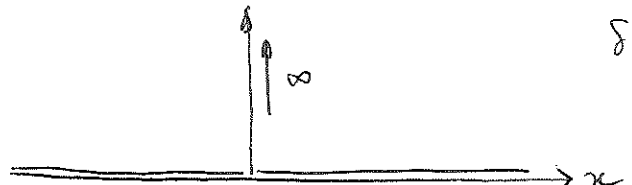
$\therefore$ The integral of the first derivative of a discontinuous function makes sense.

Notation: $u(x_0^+) - u(x_0^-) =: [\![u]\!]_{x_0}$ · Jump of u at x_0.

The situation is the same as for the Heaviside function and its derivative, the Dirac delta distribution.



$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



$$\delta(x) \text{"="} \frac{dH}{dx}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

(as a distribution)

$$\int_{-a}^{a} \frac{dH}{dx} = \int_{-a}^{a} \delta(x) = H(a) - H(-a) = 1$$

$$\int_{-a}^{a} f \frac{dH}{dx} = \int_{-a}^{a} \left[\frac{d}{dx}(fH) - H\frac{df}{dx} \right] = f(a) - \int_{0}^{a} \frac{df}{dx} = f(0)$$

However, we have:

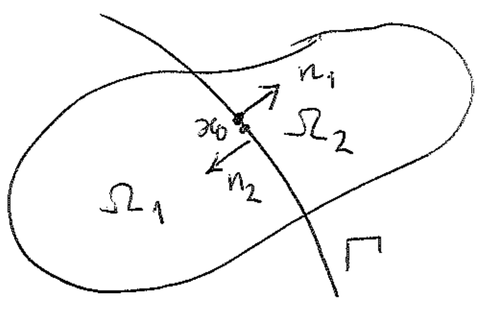
$$\int_{x_0-a}^{x_0+a} \left(\frac{du^\varepsilon}{dx} \right)^2 = \int_{x_0-a}^{x_0-\varepsilon/2} \left(\frac{du}{dx} \right)^2 + \varepsilon \left(\frac{\left(u(x_0+\varepsilon/2) - u(x_0-\varepsilon/2)\right)^2}{\varepsilon^2} \right) + \int_{x_0+\varepsilon/2}^{x_0+a} \left(\frac{du}{dx} \right)^2 \xrightarrow{\varepsilon \to 0} \infty$$

(Similarly, $\int_{-a}^{a} \delta^2 = \infty$).

Therefore, if a function is H^1 in 1D, it MUST be continuous. In 2D, it must be continuous across a curve, but can be discontinuous at an isolated point since it can be that the discontinuity has not enough weight in the integral to make it diverge. In 3D, a H^1-function must be continuous across surfaces:

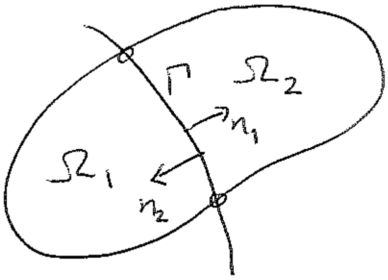
$$\boxed{[\![u]\!]_{\Gamma} = 0}$$

First transmission condition.



$$[\![u(x_0)]\!] = \lim_{\varepsilon \to 0} \left[u(x_0 + n_1 \varepsilon) - u(x_0 + n_2 \varepsilon) \right]$$

Continuity of fluxes (normal fluxes)



$\Omega = \Omega_1 \cup \Omega_2$

We have seen that

$$\int_\Omega k \nabla \delta u \cdot \nabla u = \int_\Omega \delta u f$$

$\forall \delta u \mid \delta u = 0$ on $\partial\Omega$.

On the other hand:

$$\int_{\Omega_1} \delta u (-\nabla \cdot k \nabla u) = \int_{\Omega_1} \delta u f$$

$$\int_{\Omega_1} \delta u (-\nabla \cdot k \nabla u) = -\int_{\Omega_1} \nabla \cdot (k \delta u \nabla u) + \int_{\Omega_1} k \nabla \delta u \cdot \nabla u$$

$$= -\int_{\partial\Omega_1, \Gamma} (n_1 \cdot \nabla u) k \delta u + \int_{\Omega_1} k \nabla \delta u \cdot \nabla u$$

Similarly for Ω_2. Adding up the results:

$$\int_{\Omega_1} \delta u (-\nabla \cdot k \nabla u) + \int_{\Omega_2} \delta u (-\nabla \cdot k \nabla u)$$

$$= -\int_\Gamma n_1 \cdot \nabla u k \delta u + \int_{\Omega_1} k \nabla \delta u \cdot \nabla u - \int_\Gamma n_2 \cdot \nabla u k \delta u + \int_{\Omega_2} k \nabla \delta u \cdot \nabla u$$

$$= \int_\Omega k \nabla \delta u \cdot \nabla u - \int_\Gamma \left(n_1 \cdot \nabla u k \big|_1 + n_2 \cdot \nabla u k_2 \big|_2 \right) \delta u$$

since the integral is additive and $[\![\delta u]\!]_\Gamma = 0$. Thus, since the integral of the LHS must be equal to the 1st term:

$$\boxed{\int_\Gamma \left(k_1 \frac{\partial u}{\partial n_1} + k_2 \frac{\partial u}{\partial n_2} \right) \delta u = 0} \qquad \forall \delta u \in H^1(\Omega), \ \delta u = 0 \text{ on } \partial\Omega$$

Definitions: Flux: $q = -k \nabla u$

Normal flux: $n \cdot q = -k n \cdot \nabla u = -k \dfrac{\partial u}{\partial n}$

∴ The normal fluxes must be weakly continuous.

Let $n = n_1 = -n_2$. Since the previous condition holds $\forall \delta u$, if u is regular enough:

$$[\![k \frac{\partial u}{\partial n}]\!]_\Gamma = 0 .$$

Therefore, the transmission conditions obtained for Poisson's problem are:

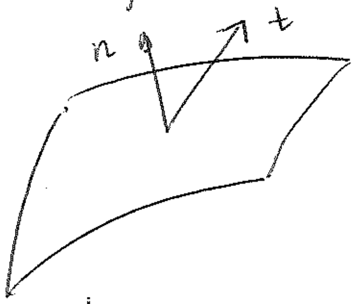
$$\boxed{\begin{aligned} &[\![u]\!]_\Gamma = 0 \\ &\int_\Gamma \delta u \left(k_1 \frac{\partial u}{\partial n_1} + k_2 \frac{\partial u}{\partial n_2} \right) = 0 \quad \forall \delta u \end{aligned}}$$

2. THE EQUATIONS OF CONTINUUM MECHANICS

In an Eulerian frame of reference, the equations of continuum mechanics are:

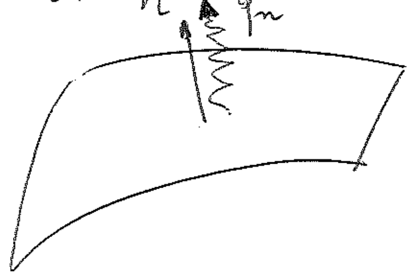
$$\left.\begin{aligned} \frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{v} &= 0 \\ \rho \frac{d\underline{v}}{dt} - \underline{\nabla} \cdot \underline{\underline{\sigma}} &= \rho \underline{b} \\ \rho \frac{de}{dt} + \underline{\nabla} \cdot \underline{q} &= r \end{aligned}\right\} \quad \begin{aligned} &\text{Conservation of mass} \\ &\quad \text{" of momentum} \\ &\quad \text{" of energy} \end{aligned}$$

ρ: density, v: velocity, σ: Cauchy stress, $\underline{b}$: body forces
$\underline{q}$: heat flux, r: heat sources.



$\underline{t} = \underline{n} \cdot \underline{\underline{\sigma}}$

Force per unit surface



$q_n = \underline{n} \cdot \underline{q}$

Heat flux per unit surface

$$\frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \underline{v} \cdot \underline{\nabla}(\cdot)$$

- In fluids, $\underline{\underline{\sigma}}$ is a function of the strain rate tensor

$$\underline{\underline{d}} = \underline{\nabla}^S \underline{v} = \frac{1}{2}\left[\underline{\nabla}\underline{v} + (\underline{\nabla}\underline{v})^t\right]$$

In the case of a Newtonian fluid:

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\mu\,\underline{\nabla}^S\underline{v} + \lambda\,\underline{\nabla}\cdot\underline{v}\,\underline{\underline{I}}$$

μ, λ: viscosity coefficients

p: thermodynamic pressure

- In solids, $\underline{\underline{\sigma}}$ is a function of either $\underline{\underline{E}}$ or $\underline{\underline{e}}$, where

• $\underline{\underline{E}} = \frac{1}{2}\left[\underline{\nabla}\underline{U} + (\underline{\nabla}\underline{U})^t + (\underline{\nabla}\underline{U})\cdot(\underline{\nabla}\underline{U})^t\right]$

Green - Lagrange strain tensor, $\underline{U}$ Lagrangian displacement

• $\underline{\underline{e}} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^t - (\underline{\nabla}\underline{u})\cdot(\underline{\nabla}\underline{u})^t\right]$

Almansi strain tensor, $\underline{u}$ Eulerian displacement.

In most materials, the constitutive law for the heat flux is

$$\underline{q} = -k\,\underline{\nabla}T \quad , \quad k \geq 0 \text{: conduction coefficient.}$$

To fix ideas, consider a fluid. The governing equations are:

$$\left.\begin{aligned} &\frac{d\rho}{dt} + \rho\,\underline{\nabla}\cdot\underline{v} = 0 \\ &\rho\frac{d\underline{v}}{dt} - \underline{\nabla}\cdot\underline{\underline{\sigma}}(\underline{v}, p) = \rho\underline{b} \\ &\rho\frac{de}{dt} - \underline{\nabla}\cdot k\underline{\nabla}T = r \end{aligned}\right\}$$

Unknowns:

$\rho, \underline{v}, e, T, p$

Closing equations:

$f(\rho, p, T) = 0$. State equation

$e = e(p, T)$. Caloric equation of state.

3. TRANSMISSION CONDITIONS

We may proceed exactly as for Poisson's problem:

$$\forall \delta\rho \quad \int_{\Omega} \delta\rho \left(\frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{v} \right) = 0$$

$$= \int_{\Omega_1} \delta\rho (\quad) + \int_{\Omega_2} \delta\rho (\quad) \quad \text{is automatically satisfied.}$$

$$\forall \delta\underline{v} \quad \int_{\Omega} \delta\underline{v} \cdot \left[\rho \frac{d\underline{v}}{dt} - \underline{\nabla} \cdot \underline{\underline{\sigma}} - \rho \underline{b} \right] = 0$$

$$= \int_{\Omega_1} \delta\underline{v} \cdot [\quad] + \int_{\Omega_2} \delta\underline{v} [\quad]$$

Note that:

$$\int_{\omega} \delta\underline{v} \cdot (-\underline{\nabla} \cdot \underline{\underline{\sigma}}) = \int_{\omega} \delta v_i \cdot (-\partial_j \sigma_{ji})$$

$$= \int_{\omega} - \partial_j (\delta v_i \sigma_{ji}) + \int_{\omega} \partial_j \delta v_i \, \sigma_{ji}$$

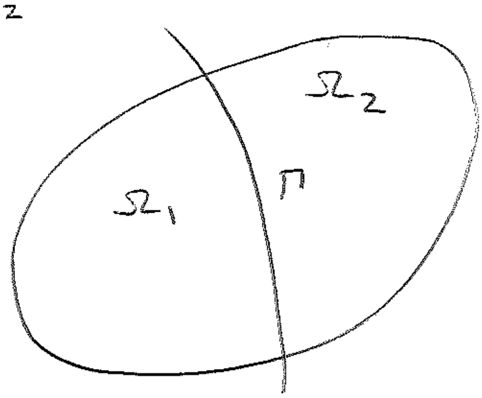
$$= -\int_{\partial\omega} \delta v_i \, n_j \sigma_{ji} + \int_{\omega} \frac{1}{2} (\partial_j \delta v_i + \partial_i \delta v_j) \cdot \sigma_{ji}$$

since $\sigma_{ij} = \sigma_{ji}$ ($\underline{\underline{\sigma}}$ is symmetric). Thus

$$\int_{\omega} \delta\underline{v} \cdot (-\underline{\nabla} \cdot \underline{\underline{\sigma}}) = -\int_{\partial\omega} \delta\underline{v} \cdot (\underline{n} \cdot \underline{\underline{\sigma}}) + \int_{\omega} \underline{\nabla}^s \delta\underline{v} : \underline{\underline{\sigma}}$$

$$\int_{\Omega} \delta\underline{v} \cdot (-\underline{\nabla} \cdot \underline{\underline{\sigma}}) = \int_{\Omega_1} \delta\underline{v} \cdot (-\underline{\nabla} \cdot \underline{\underline{\sigma}}) + \int_{\Omega_2} \delta\underline{v} \cdot (-\underline{\nabla} \cdot \underline{\underline{\sigma}})$$

$$\Rightarrow \int_{\partial\Omega} - \delta\underline{v} \cdot (\underline{n} \cdot \underline{\underline{\sigma}}) + \int_{\Omega} \underline{\nabla}^s \delta\underline{v} : \underline{\underline{\sigma}}$$



$$= \int_{\partial\Omega_1 \cap \partial\Omega} - \delta\underline{v} \cdot (\underline{n} \cdot \underline{\underline{\sigma}}) + \int_{\Omega_1} \underline{\nabla}^s \delta\underline{v} : \underline{\underline{\sigma}} + \int_{\Gamma} - \delta\underline{v} \cdot (\underline{n}_1 \cdot \underline{\underline{\sigma}}_1)$$

$$+ \int_{\partial\Omega_2 \cap \partial\Omega} - \delta\underline{v} \cdot (\underline{n} \cdot \underline{\underline{\sigma}}) + \int_{\Omega_2} \underline{\nabla}^s \delta\underline{v} : \underline{\underline{\sigma}} + \int_{\Gamma} - \delta\underline{v} \cdot (\underline{n}_2 \cdot \underline{\underline{\sigma}}_2)$$

$$\Rightarrow \boxed{\int_\Gamma \delta \underline{v} \cdot (\underline{n}_1 \cdot \underline{\underline{\sigma}}_1 + \underline{n}_2 \cdot \underline{\underline{\sigma}}_2) = 0} \quad \forall \delta \underline{v} \qquad (1)$$

Exactly as for Poisson's problem it can be argued that

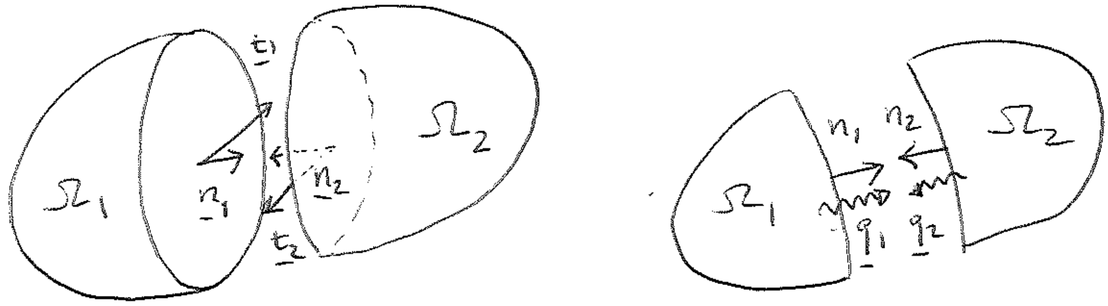
$$\boxed{[\![\underline{v}]\!]_\Gamma = 0 \quad \text{or} \quad [\![\underline{u}]\!]_\Gamma = 0} \quad \text{(equivalent)}$$

For the heat equation it is found that

$$\int_\Gamma \delta T \, (\underline{n}_1 \cdot q_1 + \underline{n}_2 \cdot \underline{q}_2) = 0 \quad \forall \delta T \quad (2)$$

$$[\![T]\!]_\Gamma = 0 \, .$$

Conditions (1) and (2) come from analytical reasoning (the additive property of the integral). However, from the physical standpoint they can also be considered <u>equilibrium</u> conditions.



$\underline{t}_1 + \underline{t}_2 = 0$ Weakly $\qquad$ $\underline{n}_1 \cdot \underline{q}_1 + \underline{n}_2 \cdot \underline{q}_2 = 0$ Weakly

<u>Remark on regularity</u> Consider the heat equation. As for Poisson's problem, $T \in H^1(\Omega)$. Its trace on Γ belongs to $H^{1/2}(\Gamma)$, and $[\![T]\!]_\Gamma = 0$ makes sense (the jump is zero "almost everywhere"). However, $\underline{q} = -k \underline{\nabla} T \in L^2(\Omega)^d$. Its normal trace is not defined as a function, but as a distribution:

$$\int_\Gamma \underbrace{\delta \cdot T}_{\in H^{1/2}(\Gamma)} \underbrace{(\underline{n} \cdot \underline{q})}_{\in H^{-1/2}(\Gamma)} < \infty$$

$H^{-1/2}(\Gamma)$, a space of distributions