ADVANCED FLUID MECHANICS
Master of Science in Computational Mechanics/Numerical Methods
Fall Semester 2018

Homework 2: Navier-Stokes and Boundary Layer

Due date: December 19, 2018

Groups of two

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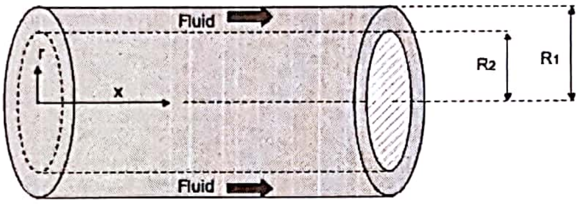
1. Consider the steady laminar flow through the annular space formed by two coaxial tubes. The radius of the outer tube is R_1 and the radius of the inner tube is R_2. The flow is along the axis of the tubes and maintained by a constant pressure gradient $\dfrac{dp}{dx}$, where the x direction is taken along the axis of the tubes.



   a) Write down the equations governing the flow motion. Clearly show the simplifications that can be done.

   b) State proper boundary conditions to be able to solve the problem.

   c) Compute the velocity at any point of the fluid in terms of the pressure gradient $\dfrac{dp}{dx}$, the fluid viscosity μ and the tubes radii R_1 and R_2. Determine the radius at which the maximum velocity is reached.

   d) Compute the volume flow rate. Express it in terms of of the pressure gradient $\dfrac{dp}{dx}$, the fluid viscosity μ, the outer tube radius R_1 and the ratio $\Phi = \dfrac{R_2}{R_1}$

   e) Consider volume flow for the limit case $\Phi \to 0$. Does the relation of (d) reduce to the formula for Poiseuille flow in a circular pipe of radius R_1? Discuss your answer.

2. Use the Kármán-Pohlhausen approximation to compute the boundary layer solution for an uniform flow over a flat plate. Assume a quadratic polynomial form for the velocity profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

and use the following boundary conditions:

$$u = 0 \text{ at } y = 0$$

$$u = U,\ \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta$$

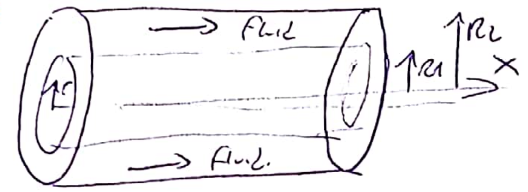
Compare this solution with Blasius exact solution.

Homework 2

Advanced Fluid Dynamics

Marcos Boniquet
(online student)

1)-



- steady flow.
- laminar flow
- $\frac{\partial P}{\partial x} = ct.$

a) Governing equations.

Assuming incompressible fluid and Newtonian fluid, we can use cylindrical coordinates Navier–Stokes equations.

mass continuity
$$\left\{ \frac{1}{r}\frac{\partial}{\partial r}(r\cancel{V_r}) + \frac{1}{r}\frac{\partial \cancel{V_\theta}}{\partial \theta} + \frac{\partial \cancel{V_x}}{\partial \cancel{x}} = 0 \right.$$

Automatically verified, given that a steady flow, with a constant section in this case implies:

$$V = (\cancel{V_r}, \cancel{V_\theta}, V_x)$$
$$V = V_x(r, \cancel{\theta}, \cancel{x}) = \boxed{V_x(r)}$$

momentum equations

$$r: \quad 0 = -\frac{\partial P^*}{\partial r}$$
$$\theta: \quad 0 = -\frac{1}{r}\frac{\partial P^*}{\partial \theta}$$

$P = P(x)$, yet we already knew $\frac{\partial P}{\partial x} = ct \longrightarrow P_{(x)} = C_1 x + \overset{P_0}{C_2}$

$$x: \quad 0 = -\frac{\partial P^*}{\partial x} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) + \frac{1}{r^2}\overset{0}{\cancel{\frac{\partial^2 V_x}{\partial \theta^2}}} + \overset{0}{\cancel{\frac{\partial^2 V_x}{\partial z^2}}}\right)$$

$$\Downarrow$$

$$\boxed{\frac{1}{\mu}\frac{\partial P^*}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)}$$

b) Boundary conditions: $\boxed{V_x(R_2) = V_x(R_1) = 0}$

c) Compute velocity $V_x \sim R_1, R_2, \mu, \frac{\partial P}{\partial x}$ and radius at which is maximum.

Using the equation $\frac{1}{\mu}\frac{\partial P^*}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) \xrightarrow{\text{integrating}} \int \frac{1}{\mu}\frac{\partial P}{\partial x} r \, dr = r\frac{\partial V_x}{\partial r} \rightarrow \frac{1}{2\mu}\frac{\partial P}{\partial x}r^2 + C_1 = r\frac{\partial V_x}{\partial r}$

$$\downarrow$$

$$\frac{1}{2\mu}\frac{\partial P}{\partial x}r + \frac{C_1}{r} = \frac{\partial V_x}{\partial r}$$

$\downarrow$ integrating again:

$$V_x(r) = \int\left(\frac{1}{2\mu}\frac{\partial P^*}{\partial x}r + \frac{C_1}{r}\right)dr$$

$$\boxed{V_x(r) = \frac{1}{4\mu}\frac{\partial P^*}{\partial x}r^2 + C_1 \ln r + C_2}$$

Using boundary conditions:
$$\left\{\begin{array}{l} \frac{R_1^2}{4\mu}\left(\frac{\partial P^*}{\partial x}\right) + C_1 \ln R_1 + C_2 = 0 \\ \frac{R_2^2}{4\mu}\left(\frac{\partial P^*}{\partial x}\right) + C_1 \ln R_2 + C_2 = 0 \end{array}\right\} \rightarrow \frac{1}{4\mu}\frac{\partial P^*}{\partial x}(R_1^2 - R_2^2) + C_1(\ln R_1 - \ln R_2) = 0$$

$$C_1 = \frac{\frac{\partial P^*}{\partial x}(R_2^2 - R_1^2)}{4\mu \ln\left(\frac{R_1}{R_2}\right)} \qquad C_2 = \frac{-R_1^2}{4\mu}\frac{\partial P^*}{\partial x} - \frac{\frac{\partial P}{\partial x}(R_2^2 - R_1^2)\ln R_1}{4\mu \ln\frac{R_1}{R_2}}$$

$$\boxed{V_x = \frac{1}{4\mu}\frac{\partial P^*}{\partial x}\boxed{\frac{(R_2^2 - R_1^2)}{\ln\left(\frac{R_1}{R_2}\right)}\ln\frac{r}{R_1} + r^2 - R_1^2}}$$

for a maximum velocity, $\boxed{\frac{\partial V_x}{\partial r} = 0}$ ← maximum.

$$\frac{\partial V_x}{\partial r} = \frac{1}{4\mu}\frac{dp^*}{dx}\left[\frac{(R_2^2 - R_1^2)}{\ln\frac{R_1}{R_2}}\frac{1}{r_{max}} + 2r_{max}\right] = 0$$

$$\frac{R_2^2 - R_1^2}{\ln\left(\frac{R_1}{R_2}\right)} + 2r^2 = 0$$

$$r_{max}^2 = \frac{R_1^2 - R_2^2}{2\ln\left(\frac{R_1}{R_2}\right)} \rightarrow r_{max} = \boxed{\sqrt{\frac{R_1^2 - R_2^2}{2\ln\left(\frac{R_1}{R_2}\right)}}}$$

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d) Compute the volume flow rate $Q \sim \frac{dp}{dx}$, μ, R_1, $\Phi = \frac{R_2}{R_1}$ → $R_2 = \Phi R_1$

$$Q = 2\pi\int_{R_1}^{R_2} V_x(r)\cdot r\,dr = \frac{\pi}{2\mu}\frac{dp^*}{dx}\left[\frac{R_1^2(\Phi-1)}{\ln\left(\frac{1}{\Phi}\right)}\int_{R_1}^{R_1\Phi} r\ln\left(\frac{r}{R_1}\right)dr + \int_{R_1}^{R_1\Phi} r^3 - R_1^2\int_{R_1}^{R_1\Phi} r\right]$$

$$Q = \frac{\pi}{2\mu}\frac{dp^*}{dx}\left[\frac{R_1^2(\Phi-1)}{-\ln\Phi}\int_{R_1}^{R_1\Phi} r\ln\left(\frac{r}{R_1}\right)dr + \left[\frac{r^4}{4}\right]_{R_1}^{R_1\Phi} - \left[\frac{R_1^2 r^2}{2}\right]_{R_1}^{R_1\Phi}\right]$$

Parts integral

$$\int_{R_1}^{R_1\Phi} r\ln\left(\frac{r}{R_1}\right) = \left[\ln\left(\frac{r}{R_1}\right)\frac{r^2}{2}\right]_{R_1}^{R_1\Phi} - \int_{R_1}^{R_1\Phi}\frac{r^2}{2}\frac{1}{r}dr = \left[\ln\left(\frac{r}{R_1}\right)\frac{r^2}{2} - \frac{r^2}{4}\right]_{R_1}^{R_1\Phi} = \frac{r^2}{2}\left[\ln\left(\frac{r}{R_1}\right) - \frac{1}{2}\right]_{R_1}^{R_1\Phi} = \frac{R_1^2}{2}\Phi\ln\Phi$$

$u = \ln\frac{r}{R_1}$ $du = \frac{1}{r}$

$dv = r$ $v = \frac{r^2}{2}$

$$Q = \frac{\pi}{2\mu}\frac{dp^*}{dx}\left[\frac{R_1^2(1-\Phi)}{\cancel{\ln\Phi}}\cdot\frac{R_1^2}{2}\Phi\cancel{\ln\Phi} + \frac{R_1^4}{4}(\Phi^4-1) + \frac{R_1^4}{2}(\Phi^2-1)\right]$$

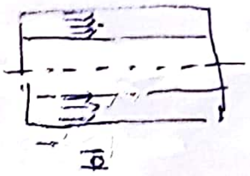
$$\boxed{Q = \frac{\pi}{2\mu}\frac{dp^*}{dx}R_1^4\left[\frac{(1-\Phi)\Phi}{2} + \frac{(\Phi^4-1)}{4} + \frac{(\Phi^2-1)}{2}\right]}$$

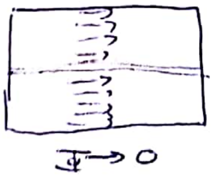
e) Consider $\Phi \to 0$ (this is $R_1 \ggg R_2$). Does this led to Poiseuille formula for circular pipe?? of radius R_1?

$$\underset{\Phi\to 0}{Q} = \frac{\pi}{2\mu}\frac{dp^*}{dx}R_1^4\left(0 - \frac{1}{4} + \frac{1}{2}\right) = \frac{\pi}{8\mu}\frac{dp^*}{dx}R_1^4 \underset{R_1 = \frac{D}{2}}{=} \frac{\pi}{8\mu}\frac{dp^*}{dx}\frac{D^4}{16} = \boxed{\frac{\pi D^4}{128\mu}\frac{dp^*}{dx}}$$

it is EXACTLY Poiseuille's formula for circular pipe!

This makes sense, given that "interior" tube becomes negligible.





Almost a completely full cylinder. Inside cylinder negligible

2

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2)- Use Karman-Pohlhausen to compute the boundary layer solution for on uniform flow over a flat plate.

Assume a quadratic polynomial form: $\boxed{\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2}$

use the following boundary conditions: $\left\{\begin{array}{l} u=0 \quad \text{if } y=0 \\ \left.\begin{array}{l} u=U \\ \frac{\partial u}{\partial y}=0 \end{array}\right\} \text{if } y=\delta \end{array}\right.$

$y=0 \; : \; \frac{u}{U} = a \rightarrow \boxed{a=0}$

$y=\delta \quad \frac{U}{U} = b+c \rightarrow b+c=1$

$\frac{1}{U}\underset{0}{\underset{\parallel}{\left(\frac{\partial u}{\partial y}\right)}} = \frac{b}{\delta} + \frac{2c}{\delta^2}\overset{\boxed{y=\delta}}{y} \rightarrow 0 = \frac{b}{\delta} + \frac{2c}{\delta^2}\delta$

$c = 1-b$

$0 = b + 2(1-b)$

$0 = b + 2 - 2b$

$\boxed{b=2}$ $\boxed{c=-1}$

$\boxed{\begin{array}{l} a=0 \\ b=2 \\ c=-1 \end{array}}$

Resulting in $\boxed{\boxed{\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2}}$

Comparison with Blasius solution:

$\left\{\begin{array}{l} u = U f' \\ v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,\eta f' - \frac{1}{2}\sqrt{\frac{\nu U}{x}}\cdot f \end{array}\right.$

$f = f\Bigg(\overset{\eta}{\frac{y}{\sqrt{\nu x/U}}}\Bigg)$

PDE

$\boxed{f''' + \frac{1}{2}ff'' = 0}$

Boundary conditions $\left\{\begin{array}{l} f(0) = f'(0) = 0 \\ f'(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty \rightarrow f'(\infty) = 1 \end{array}\right.$

$\frac{u}{U} = a + b\eta\frac{\sqrt{\nu x/U}}{\delta} + \frac{c\eta^2 \nu x}{\delta^2 U} = f'(\eta)$

$y = \eta\sqrt{\nu x/U}$

$\boxed{f'(\eta=0)=0} \rightarrow \boxed{\boxed{a=0}}$

$\cancel{a}^{c}+ \quad \left(b\frac{\sqrt{\nu x/U}}{\eta\delta} + \frac{c\,\nu x}{\delta^2 U}\right) = \frac{f'}{\eta^2}$

$\boxed{f'(\eta) \rightarrow 1 \quad \eta = \infty}$

$\infty\left(b\frac{\sqrt{\nu x/U}}{\delta} + \eta\frac{c\,\nu x}{\delta^2 U}\right) = 1$

$\boxed{\frac{c\,\nu x}{\delta^2 U} = 0} \rightarrow \boxed{\boxed{c=0}}$

$\frac{u}{U} = b\eta\frac{\sqrt{\nu x/U}}{\delta} = f'(\eta)$

again $f'(\infty) = 1$

$\downarrow$

$\boxed{\boxed{b=0}}$

$\downarrow$

$\frac{u}{U} = 0 \rightarrow u = 0$

3