ADVANCED FLUID MECHANICS

Master of Science in Computational Mechanics/ Numerical Methods

Fall Semester 2015

Homework 4: Navier-Stokes equations and Boundary Layer

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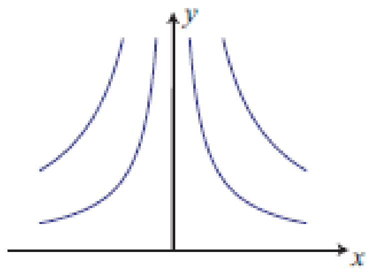
Exercise 1· Consider a fluid stream whose velocity vector coincides with the y axis that impinges in a plane boundary that coincides with the x axis, as shown in the figure below.



(a) If we consider an ideal fluid, velocity can be obtained using the following stream function

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

Compute the velocity field in Cartesian coordinates (u, v) and show that it verifies the boundary conditions. Obtain an expression for the pressure distribution.

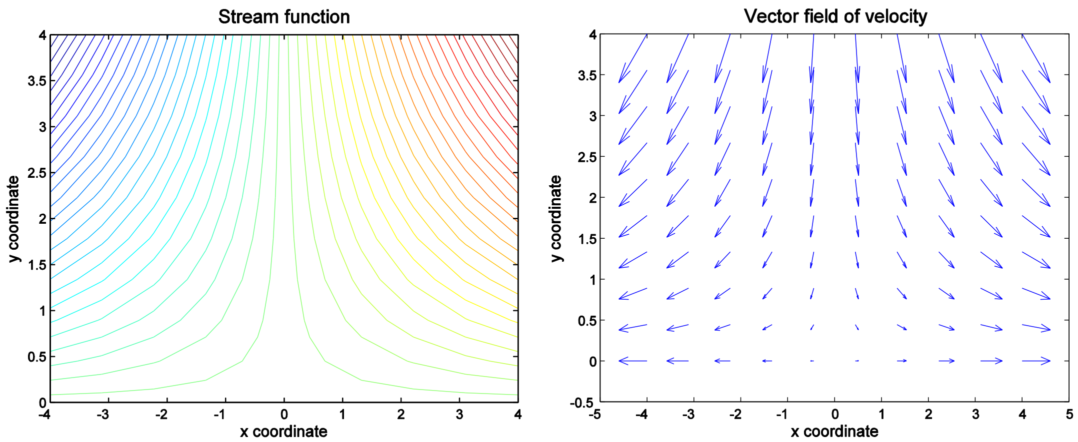


Figure 1. Stream function contours (left) and vector plot of the velocity field (right).

Taking into account that we are considering an incompressible fluid in a plane domain we can express the velocity field in terms of the stream function through the following relations,

$$u = \frac{\partial \psi}{\partial y} \; ; v = -\frac{\partial \psi}{\partial x},$$

Furthermore, we have that

$$\sin(2\theta) = 2sincos\theta \,,$$

$$x = rcos\theta; y = rsin\theta \,,$$

So, now we can express the stream function in Cartesian coordinates yielding

$$\psi(r,\theta) = Ur^2 \sin(2\theta) = U2rsin\theta rcos\theta \,,$$

$$\psi(x,y) = 2Uxy \,,$$

Being the associated velocity field,

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(2Uxy) = 2Ux \,,$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(2Uxy) = -2Uy \,,$$

So,

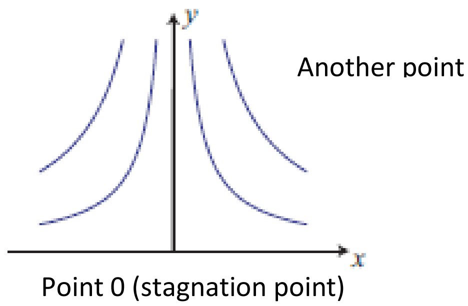
$$\boldsymbol{V} = (2Ux, -2Uy).$$

It may be shown that this velocity field satisfies the prescribed boundary conditions which are:

$$u(x,0) = u(x) \quad (slip\ condition)$$

$$v(x,0) = 0 \quad (no\ penetration)$$

Assuming inviscid flow, whether subject to conservative body forces or neglecting them, we can apply the Bernoulli principle between points 0 and another point in order to obtain the pressure distribution:



Reading,

$$\frac{1}{2}(\|\mathbf{V}\|)^2 + \frac{p}{\rho} + gz - \frac{1}{2}(\|\mathbf{V_0}\|)^2 - \frac{p_0}{\rho} - gz_0 = 0 \,,$$

Concerning this equality we know the following information:

- Whether not taking into account body forces or through the statement that, the stagnation region is very narrow, we vanish the body forces terms.
- Velocity at the stagnation point is equal to zero.

Accounting for all the aforementioned the Bernoulli equation yields,

$$\frac{1}{2}(\|\mathbf{V}\|)^2 + \frac{p}{\rho} - -\frac{p_0}{\rho} = 0\,,$$

Being,

$$\|\mathbf{V}\| = \sqrt{(2Ux)^2 + (-2Uy)^2}\,,$$

$$p(x,y) = p_0 - \rho\frac{1}{2}\left(\sqrt{(2Ux)^2 + (-2Uy)^2}\right)^2 = p_0 - 2\rho U^2(x^2 + y^2)\,. \tag{1}$$

(b) Show that the former velocity and pressure distributions verify the Navier-Stokes equations but not the boundary conditions for the viscous problem.

Mass conservation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

Momentum conservation,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho b_x\,,\quad X - Momentum \tag{3a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho b_y\,,\quad Y - Momentum \tag{3b}$$

To simplify the set of equations we apply all the information we know of the stated problem:

· Plane flow:

$$\mathrm{w} = 0\,,$$

· Stationary velocity field so:

$$\frac{\partial u}{\partial t} = 0;\ \frac{\partial v}{\partial t} = 0\,,$$

· Body forces no considered:

$$\rho b_x = 0;\ \rho b_y = 0\,,$$

· Potential flow:

$$\nabla^2\boldsymbol{V} = \boldsymbol{0}\,,$$

And further simplifications due to derivation of velocity field in equations (3a) and (3b), these ones simplifies into,

$$\rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \tag{4a}$$

$$\rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} \tag{4b}$$

In order to demonstrate equations (4a) and (4b) we have from (1),

$$p(x,y) = -2\rho U^2(x^2 + y^2),$$

And finally we get,

$$-\frac{\partial p}{\partial x} = 4\rho U^2 x = \rho u \frac{\partial u}{\partial x}$$

$$-\frac{\partial p}{\partial y} = 4\rho U^2 y = \rho v \frac{\partial v}{\partial y}$$

However in this case the non-slip boundary conditions is not accomplished whereas the no penetrability condition is.

$$u(x,0) \neq 0 \quad (non - slip\ condition)$$

$$v(x,0) = 0 \quad (no\ penetrability),$$

(c) A solution for the viscous problem can be obtained modifying the potential fow in such a way that meeting the boundary conditions would be possible. If we attempt,

$$u = 2Uxf'(y)$$

Show that the continuity equation requires, $v = -2Uf(y)$. State appropriate boundary conditions for the function *f*.

Recalling the equation of continuity (2), and applying the new definition of u,

$$\frac{\partial}{\partial x}\left(2Uxf^{'}(y)\right) + \frac{\partial v}{\partial y} = 0\ ,$$

proceeding,

$$\frac{\partial v}{\partial y} = -2Uf^{'}(y) \quad and \quad v = -2Uf(y).$$

The suitable boundary conditions for the viscous problem are first, the non-slip condition,

$$u(x,0) = 2Uxf^{'}(0) = 0 \quad so \quad f^{'}(0) = 0 \tag{5}$$

$$v(x,0) = -2Uf(0) = 0 \quad so \quad f(0) = 0 \tag{6}$$

And in addition it must be considered that at the far-field we recover the potential flow expression so,

$$u(x,y) = u_{potential} \qquad as\ y \to \infty$$

So under this consideration we know that,

$$u = 2Ux = 2Uxf^{'}(y) \quad what\ implies\ that\ f^{'}(y) = 1 \tag{7}$$

(d) Use the y-momentum equation to obtain an expression for the pressure distribution in terms of the function f. In order to completely determine the pressure distribution, you can use that for a large value of y, the potential flow pressure should be recovered.

We recall the y-momentum equation (3b), and after simplifying it turns into,

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\nabla^2 v\,,$$

Considering that,

$$u=2Uxf'(y)=2Uxf'\,,$$

$$v=-2Uf(y)=-2Uf\,,$$

$$\rho\left((2Uxf')\frac{\partial}{\partial x}(-2Uf)+(-2Uf)\frac{\partial}{\partial y}(-2Uf)\right)=-\frac{\partial p}{\partial y}+\mu\nabla^2(-2Uf),$$

After working with the equation we get to,

$$\frac{\partial p}{\partial y}=-4U^2\rho ff'-2\rho U\nu f''\,,$$

And integrating respect to y we obtain,

$$p(x,y)=-2U^2\rho(f')^2-2\rho U\nu f'+g(x)\,,$$

But as it is stated in the problem we know that,

$$if\ y\to\infty\quad f(y)=y\ and f'(y)=1\quad so\ at\ the\ far\ field$$

$$p(x,y)=-2U^2\rho y^2-2\rho U\nu+g(x)\,,$$

At this point we recover the expression of the pressure distribution for a potential flow obtained previously and in order to know g(x) we equal both expressions

$$p(x,y)=-2\rho U^2(x^2+y^2)=-2U^2\rho y^2-2\rho U\nu y+g(x),$$

and,

$$g(x)=p_0-2\rho U^2x^2+2\rho U\nu\,,$$

The pressure distribution once the potential flow solutions has been modified reads,

$$p(x,y)=p_0-2\rho U^2(f^2+x^2)+2\rho U\nu(1-f').$$

(e) Using the x-momentum equation and the pressure distribution obtained in the previous point, obtain a differential equation for the function f. Show that the problem can be solved using the boundary conditions stated in point c).

Now we recall the expression of the x-momentum balance (3a) as well as, the expression for the pressure distribution obtained in c) so,

$$-\frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}\left(p_0 - 2\rho U^2(f^2 + x^2) + 2\rho U\nu(1 - f')\right) = 4\rho U^2 x^2 ,$$

$$\mu\nabla^2 u = \frac{\partial^2}{\partial y^2}(2Uxf') = 2U\nu x f''' ,$$

And after operating with the u and v modified velocity fields for the viscous problem we get to,

$$\frac{\nu}{2U} f''' + ff'' - (f')^2 + 1 = 0 .$$

At this point we recall the boundary conditions stated previously (5-7). But it would be more appropriate to solve a problem which didn't take into consideration a particular viscosity but, instead of this that would be valid for every kinematic viscosity we wanted, independently of the fluid considered. In order to accomplish this it is necessary to do a change of variable, which reads,

$$\varphi(\delta) = \sqrt{\frac{2U}{\nu}} f(y), \quad and \quad \delta = \sqrt{\frac{2U}{\nu}} y$$

So now the problem can be solved in terms of$\varphi(\delta)$, satisfying the following O.D.E. and its boundary conditions,

$$\varphi''' + \varphi\varphi'' - (\varphi')^2 + 1 = 0, \qquad (8)$$

$$\varphi(0) = \varphi'(0) = 0,$$

$$\varphi'(\delta) = 1 \quad when \quad \delta \to \infty ,$$

Now we have a third order non-linear O.D.E. with three boundary conditions so we solve can solve it.

To do so, first we apply the following change of variables in equation (8) in order to obtain a system of first order O.D.E's:

$$\varphi_1 = \varphi; \;\; \varphi_2 = \varphi_1' \; ; \; \varphi_3 = \varphi_2'$$

Obtaining the system

$$\begin{cases} \varphi_3' = -\varphi_1\varphi_3 + \varphi_2\varphi_2 + 1 \\ \varphi_2' = \varphi_3 \\ \varphi_1' = \varphi_2 \end{cases}$$

Finally we can solve the boundary value problem stated above as a initial value problem, using the shooting method with trapezoidal method, guessing the value of $\varphi_3(0) = L$ such that $\varphi_2(\infty) = 1$.

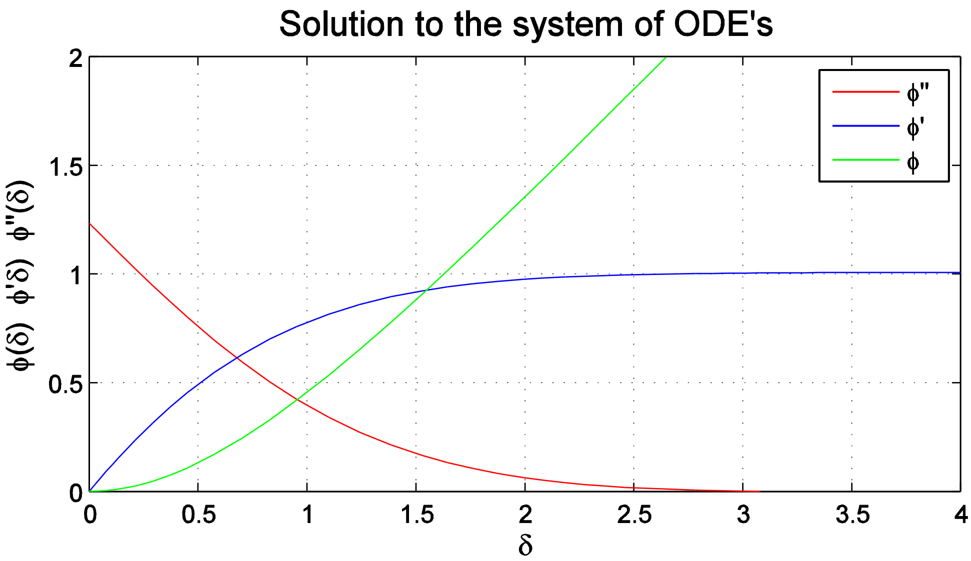


Figure 2. Solution for the equivalent system of ODE's

Exercise 2·Use the Kárman-Pohlhausen approximation to compute the boundary layer solution for an uniform flow over a flat plate. Assume a quadratic form for the velocity profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

And use the following boundary conditions:

$$u = 0 \; at \; y = 0$$

$$u = U, \frac{\partial u}{\partial y} = 0 \; at \; y = \delta$$

Compare the results with the exact Blausius solution and with the ones obtained assuming a cubic velocity profile.

Defining a similarity variable η as $\eta = y/\delta$, the velocity profile is transformed into:

$$\frac{u}{U} = a + b\eta + c\eta^2$$

And the boundary conditions into:

$$\frac{u}{U} = 0 \; at \; \eta = 0$$

$$\frac{u}{U} = 1, \qquad \frac{\partial \, u/U}{\partial \eta} = 0 \; at \; \eta = 1$$

Applying the boundary conditions we can solve for the 3 coefficients a, b, and c as follows

$$\begin{cases} 0 = a + b \cdot 0 + c \cdot 0^2 \\ 1 = a + b \cdot 1 + c \cdot 1^2 \\ 0 = b + c\eta \end{cases} \xrightarrow{solving} \begin{cases} a = 0 \\ b = 2 \\ c = -1 \end{cases}$$

Finally the profile is of the form:

$$\frac{u}{U} = 2\eta - \eta^2 \tag{9}$$

Over a flat plate $U\frac{dU}{dx} = 0$, and therefore the Kármán momentum integral equation reduces to

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_0}{\rho}$$

Where $U^2\theta$ is the momentum thickness computed as

$$U^2\theta = \int_0^\delta u(U - u)dy$$

Moving U^2 to the r.h.s, and transforming the integral into terms of the similarity variable ($dy \ = \delta d\eta, \ and \ y = \delta -> \eta = 1$)

$$\theta = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)\delta d\eta$$

Computing using (9)

$$\theta = \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\delta d\eta = \frac{2}{15}\delta$$

In the reduced Kármán momentum integral equation the r.h.s. is computed as:

$$\frac{\tau_0}{\rho} = \nu\frac{\partial u}{\partial y}\bigg|_{y=0} \rightarrow \frac{\tau_0}{\rho} = U\nu\frac{\partial u/U}{\delta\partial\eta}\bigg|_{\eta=0} = \frac{2U\nu}{\delta}$$

Reaching the final expression

$$\frac{d}{dx}\left(U^2\frac{2}{15}\delta\right) = \frac{2U\nu}{\delta}$$

$$\delta \ d\delta = \frac{2U\nu}{U^2\frac{2}{15}}dx$$

Which integrating gives:

$$\frac{\delta^2}{2} = \frac{2U\nu}{U^2\frac{2}{15}}x + C \xrightarrow{\delta(0)=0} C = 0$$

$$\delta = \sqrt{\frac{30\nu x}{U}} = \frac{5.48}{\sqrt{Re_x}}x$$

With $Re_x = Ux/\nu$

Finally computing θ/x

$$\frac{\theta}{x} = \frac{2\delta}{15x} = \frac{2\frac{5.48}{\sqrt{Re_x}}x}{15x} = \frac{0.73}{\sqrt{Re_x}}$$

From the results available in the lecture notes we can compare the results of the Kármán-Pohlhausen approximation for a quadratic profile, with the ones assuming a cubic profile and the exact Blausius solution.

	δ/x	θ/x
Kármán-Pohlhausen with quadratic profile	$\frac{5.48}{\sqrt{Re_x}}$	$\frac{0.73}{\sqrt{Re_x}}$
Kármán-Pohlhausen with cubic profile	$\frac{4.64}{\sqrt{Re_x}}$	$\frac{0.646}{\sqrt{Re_x}}$
Exact Blausius solution	$\frac{5}{\sqrt{Re_x}}$	$\frac{0.664}{\sqrt{Re_x}}$

From the table above we can see that using Kármán-Pohlhausen approximation and assuming a cubic profile gives better results than assuming a quadratic profile. We can also see that with respect to the exact Blausius solution the cubic profile has a far more accurate solution for the momentum thickness, whereas for the displacement thickness both profiles differ more or less in the same magnitude.

Annex· Matlab files for the shooting method (Modified from [1])

File solvesys.m:

```
g1=0; g2=2;

 while((g2-g1)>10e-6)
    L=(g1+g2)/2;
    [delta,phi] = ode45('ode',[0 6],[0 0 L])
     if max(phi(:,2))>1
         g2=L;
     else
         g1=L;
     end
 end

plot(delta,phi(:,3),'r',delta,phi(:,2),'b',delta,phi(:,1),'g')
axis equal
axis([0 4 0 2])
title('Solution to the system of ODE''s', 'FontSize', 14)
xlabel('\delta', 'FontSize', 12) % x-axis label
ylabel('\phi(\delta)  \phi''\delta)  \phi''''(\delta)', 'FontSize',
12) % y-axis label
legend('\phi''''','\phi''','\phi')
grid on
```

File ode.m:

```
function df = ode(~,phi)
    df = zeros(size(phi));
    df(1) = phi(2);
    df(2) = phi(3);
    df(3) = -phi(1)*phi(3) + phi(2)*phi(2) -1;
end
```

[1] http://www.mathworks.com/matlabcentral/answers/52075-solving-a-third-order-non-linear-ode-using-ode45