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Advanced Fluid Mechanics : Homework 4

① a) Ideal fluid, stream function:

$$\psi(r,\theta) = U r^2 \cdot \sin(2\theta)$$

- Velocity field in cartesian coordinates (u,v) and showing bc. verification.
- Expression for the pressure distribution.

Velocity field in polar $\begin{cases} v_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \rightarrow v_r = 2Ur\cdot\cos(2\theta) \\ v_\theta = -\dfrac{\partial\psi}{\partial r} \rightarrow v_\theta = -2Ur\cdot\sin(2\theta). \end{cases}$

Velocity field in cartesian $\begin{cases} u = u_r\cdot\cos\theta - u_\theta\cdot\sin\theta \\ v = u_r\cdot\sin\theta + u_\theta\cdot\cos\theta \\ x = r\cdot\cos\theta,\quad y = r\cdot\sin\theta,\quad r = \sqrt{x^2+y^2} \end{cases}$

$$\begin{cases} u = 2Ur\cdot\cos 2\theta\cdot\cos\theta + 2Ur\cdot\sin 2\theta\cdot\sin\theta \\ v = 2Ur\cdot\cos 2\theta\cdot\sin\theta - 2Ur\cdot\sin 2\theta\cdot\cos\theta \end{cases}$$

$x = r\cdot\cos\theta$
$y = r\cdot\sin\theta$

$$\begin{cases} u = 2U\cos 2\theta\, x + 2U\sin 2\theta\, y \\ v = 2U\cos 2\theta\, y - 2U\sin 2\theta\, x \end{cases}$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\sin(2\theta) = 2\sin\theta\cdot\cos\theta$
$\cos^2\theta + \sin^2\theta = 1$

$$\begin{cases} u = 2Ux\,(2\cos^2\theta - 1) + 2Uy\,(2\sin\theta\cos\theta). \\ v = -2Ux\,(2\sin\theta\cos\theta) + 2Uy\,(2\cos^2\theta - 1) \end{cases}$$

$$\begin{cases} u = -2Ux + 4U\cos\theta\,(\cos\theta\, x + \sin\theta\, y) \\ v = -2Uy + 4U\cos\theta\,(\cos\theta\, y - \sin\theta\, x) \end{cases}$$

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$u = -2Ux + 4U\left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}\right) = 2Ux$$

$$v = -2Uy + 4U\left(\frac{y}{\sqrt{x^2+y^2}}\right)\left(\frac{xy}{\sqrt{x^2+y^2}} + \frac{xy}{\sqrt{x^2+y^2}}\right) = -2Uy$$

* Velocity field (cartesian coordinates) $\Rightarrow$ $\boxed{\underline{V} = [2Ux, -2Uy]}$

* BC. verification
- $\underline{V} = 0$ in stagnation point $(0,0) \rightarrow \underline{V}(0,0) = [0,0]$ ✓
- in $x=0$, v paralel to y-axis $\rightarrow \underline{V}(0,y) = (0, -2Uy]$ ✓
- in the plate, $v=0 \rightarrow v(x,0) = [2U(x), 0]$ ✓

* Presure distribution

$$\omega = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2Ux & -2Uy & 0 \end{vmatrix} = \cancel{\frac{\partial}{\partial x}(-2Uy)e_z}^{\,0} + \cancel{\frac{\partial}{\partial z}(2Ux)e_y}^{\,0} = 0$$

Irrotational velocity field.

So the Bernoulli eq. becomes.

$$\cancel{\int_1^2 \overbrace{\frac{\partial \underline{V}}{\partial t}}^{\text{stationary problem}} d\underline{s}} + \left(\frac{1}{2}\underline{V_2}^2 + \frac{P_2}{\rho} - F_2\right) - \left(\frac{1}{2}\underline{V_1}^2 + \frac{P_1}{\rho} - F_1\right) = 0$$

$$|\underline{V}| = \sqrt{(2Ux)^2 + (-2Uy)^2} = 2U\sqrt{x^2+y^2}$$

Taking 1 as stagnation point and 2 as any other point in the domain.

$$\left(\frac{1}{2}\left(2U\sqrt{x^2+y^2}\right)^2 + \frac{P_2}{\rho} + \cancel{g y_2}\right) - \left(0 + \frac{P_{max}}{\rho} + \cancel{g y_1}\right) = 0 \qquad y_2 = y_1$$

$$2U^2(x^2+y^2) + \frac{P_2}{\rho} = \frac{P_{max}}{\rho}$$

$$\frac{P_{max} - P_2}{\rho} = 2U^2 (x^2 + y^2)$$

$$\boxed{P(x,y) = P_{max} - 2U^2 \rho (x^2+y^2)}$$

Presure at the stagnation point.

b) Navier - Stokes eq:

- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (Mass conservation).

$$\frac{\partial (2Ux)}{\partial x} + \frac{\partial (-2Uy)}{\partial y} = 0 \rightsquigarrow 2U - 2U = 0 \checkmark$$

Negligible bf.

- $\rho \left(\cancel{\frac{\partial u}{\partial t}} + u \frac{\partial u}{\partial x} + \cancel{v \frac{\partial u}{\partial y}}^{0} \right) = - \frac{\partial P}{\partial x} + \mu \nabla^2 u + \cancel{\rho b_x}$ (x - momentum).

STEADY State

$$\rho (2Ux) \frac{\partial (2Ux)}{\partial x} = - \frac{\partial (P_{max} - 2U^2 \rho (x^2+y^2))}{\partial x} + \mu \nabla^2 (2Ux).$$

$$\rho 4U^2 x = 2U^2 \rho 2x + \mu \left(\cancel{\frac{\partial^2 (2Ux)}{\partial x^2}}^{0} + \cancel{\frac{\partial^2 (2Ux)}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 (2Ux)}{\partial z^2}}^{0} \right)$$

$$\rho 4U^2 x = 2U^2 \rho 2x \checkmark$$

- $\rho \left(\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial v}{\partial x}}^{0} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial P}{\partial x} + \mu \nabla^2 u + \rho b_y$ (y - momentum).

$$\rho (-2Uy) \frac{\partial (-2Uy)}{\partial y} = \frac{- \partial (P_{max} - 2U^2 \rho (x^2+y^2))}{\partial y} + \mu \nabla^2 (-2Uy).$$

$$\rho 4U^2 y = 2U^2 \rho 2y + \mu \cancel{\nabla^2 u}^{0}$$

$$\rho 4U^2 y = \rho 4U^2 y \checkmark$$

All the navier - Stokes equations are fullfiled

For viscous fluids velocity must be zero at the "wall" ($y=0$).

- $u = 2Ux \rightsquigarrow$ if $x \neq 0$ and $u \neq 0 \rightarrow$ No slip conditions at wall
- $v = 2Uy \rightsquigarrow$ if $y = 0$ and $v = 0 \rightarrow$ Vertical bc is fulfilled.

c)

$$u = 2Ux\, f'(y)$$
$$v = -2U\, f(y)$$

State appropiate bc for the function "f"

$\underline{V}(x,0) = 0 \longrightarrow$ Condition in the surface

$\underline{V}(x,y) = (2Ux, -2Uy) \rightarrow$ Potential flow velocity field for a point far from the surface.

$$0 = 2Ux\, f'(0) \rightsquigarrow f'(0) = 0$$
$$0 = -2U\, f(0) \rightsquigarrow f(0) = 0$$
$$2Ux = 2Ux\, f'(y) \rightsquigarrow f'(y) = 1$$
$$-2Uy = -2U\, f(y) \rightsquigarrow f(y) = y$$

d). y-momentum eq:
$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\text{Steady problem.}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \cancel{\rho b_y}$$

$$\rho\left((-2Ux\, f'(y))\,\underbrace{\cancel{\frac{\partial(-2Uf(y))}{\partial x}}}_{0} + (-2U f(y))\,\frac{\partial(-2Uf(y))}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2(-2Uf(y))$$

$$\rho\left(-2Uf(y)\left(-2U\frac{\partial f(y)}{\partial y}\right)\right) = -\frac{\partial p}{\partial y} + \mu\left(\underbrace{\cancel{\frac{\partial^2(-2Uf(y))}{\partial x^2}}}_{0} + \frac{\partial^2(-2Uf(y))}{\partial y^2}\right)$$

$$\rho\left[4U^2 f(y) f'(y)\right] = -\frac{\partial p}{\partial y} + \mu(-2U f''(y))$$

$$\int \boxed{\frac{1}{\rho}\frac{\partial p}{\partial y}dy = \int -2U\nu f''(y)\,dy \int -4U^2 f(y) f'(y)\,dy} \longleftarrow \text{Integrating.}$$

$$\int 2f(y)\, f'(y)\, dy = \left[f(y)\right]^2$$

$$\frac{P(x,y)}{\rho} = -2U\nu f'(y) - 2U^2 (f(y))^2 + P_0(x)$$

$$P(x,y) = -2U\mu f'(y) - 2\rho U^2 f^2(y) + P_0(x)$$

If $y \to \infty$ the potential flow is recovered, so the presure distribution must be the same that the one obtained in the exercice a).

- $P(x,y) = P_{max} - 2U^2\rho(x^2+y^2)$
- If $y \uparrow\uparrow \rightarrow f'(y) = 1$

$f(y) = y$

$$P(x,y) = -2U\mu\underbrace{f'(y)}_{1} - 2\rho U^2\underbrace{f^2(y)}_{y^2} + P_0(x) = P_{max} - 2U^2\rho(x^2+y^2).$$

$$-2U\mu - 2\rho U^2 y^2 + P_0(x) = P_{max} - 2U^2\rho(x^2+y^2)$$

$$\Downarrow$$

$$P_0(x) = P_{max} - 2U^2\rho x^2 + 2U\mu \rightarrow \boxed{P_0(x) = P_{max} + 2U(\mu - U\rho x^2).}$$

$$P(x,y) = P_{max} + 2U\mu(1-f'(y)) - 2U^2\rho(f^2(y)+x^2).$$

e) x- momentum eq: $\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\nabla^2 u + \cancel{\rho b_x}^{\,0}$

- $P(x,y) = P_{max} + 2U\mu(1-f'(y)) - 2U^2\rho(f^2(y)+x^2)$
- $\underline{v} = [2Ux f'(y), -2Uf(y)]$

$$\rho\left[(2Uxf'(y))\frac{\partial(2Uxf'(y))}{\partial x} + (2Uf(y))\frac{\partial(2Uxf'(y))}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\nabla^2(2Uxf'(y)).$$

$$\rho[4U^2x(f'(y))^2 - 4U^2xf(y)f''(y)] = -(-4U^2\rho x + 2Ux f'''(y)\mu$$

$$1/4U^2\rho x \cdot (4U^2\rho x(f'(y))^2 - 4U^2\rho x f(y)f''(y)) = (4U^2\rho x + 2Uxf'''(y)\mu)\cdot 1/4U^2\rho x$$

$$(f'(y))^2 - f(y)f''(y) - 1 - \frac{f'''(y)}{2U\rho}\mu = 0$$

$$(f'(y))^2 - f(y)f''(y) - 1 - \frac{\nu f'''(y)}{2U} = 0$$

$$\boxed{\frac{\nu}{2U}f'''(y) + f''(y)f(y) - f'(y)f'(y) + 1 = 0} \leftarrow \text{EDO}$$

B.C.s:

$$\left.\begin{array}{ll} f'(0)=0 & f'(y)=1 \\ f(0)=0 & f(y)=y \end{array}\right\} (y\to\infty).$$

$$\frac{\nu}{2U} f'''(0) + \cancel{f''(0)f(0)} - \cancel{f'(0)f'(0)} + 1 = 0$$

$$\frac{\nu}{2U} f'''(0) + 1 = 0 \to \boxed{f'''(0) = \frac{-2U}{\nu}}$$

$$\frac{\nu}{2U} f'''(y) + \cancel{f''(y)f(y)} - f'(y)f'(y) + 1 = 0$$

$$\frac{\nu}{2U} f'''(y) - 1 + 1 = 0 \to \boxed{f'''(y) = 0}$$

② Boundary layer solution for an uniform flow over a flat plate:

- Blasius exact solution
- Kármán - Pohlhausen approx: * Cuadratic
  * Cubic

Compare.

Kármán - Pohlhausen approximation (cuadratic).

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

$u=0$ for $y=0$ $\to$ $\frac{U}{u}=0$ for $y/\delta=0$

$u=U$; $\frac{\partial u}{\partial y}=0$ for $y=\delta$ $\to$ $\frac{u}{U}=1$; $\frac{\partial(u/U)}{\partial(y/\delta)}=0$ for $y/\delta=1$.

- $0 = a + b\cdot 0 + c\cdot 0 \to a=0$
- $1 = b(1) + c(1)^2$
- $0 = b + 2c(1)$

$$\left\{ 1 = -2c + c \to \begin{array}{l} b=2 \\ c=-1 \end{array} \right. \to \frac{u}{U} = 2\left(y/\delta\right) - \left(y/\delta\right)^2$$

Momentum integral equation (flat plate): $\frac{d}{dx}\int_0^\delta u\,(U-u)\,dy = \frac{\tau_0}{\rho}$

$$\tau_0 = \mu \frac{U}{\delta} \left.\frac{\partial(u/U)}{\partial(y/\delta)}\right|_{y/\delta=0}$$

$$\frac{d}{dx}\int_0^\delta U\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\left(U-U\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\right)dy = \frac{\tau_0}{\rho}$$

$$U^2\frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)dy = \frac{\tau_0}{\rho} \qquad \tau_0 = \mu\frac{U}{\delta}$$

$$\frac{d}{dx}\int_0^\delta \left(\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy = \frac{\nu}{\delta U}$$

$$\frac{d}{dx}\left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta = \frac{2\nu}{\delta U} \longrightarrow \frac{d}{dx}\left(\frac{\cancel{2}\delta}{15}\right) = \frac{\cancel{2}\nu}{\delta U}$$

$$\boxed{\frac{d}{dx}(\delta) = \frac{15\nu}{\delta U}} \longrightarrow \underline{\text{ODE}}$$

$$\delta\cdot d\delta = \frac{15\nu}{U}dx \rightarrow \int \delta\cdot d\delta = \int\frac{15\nu}{U}dx \rightarrow \frac{\delta^2}{2} = \frac{15\nu x}{U}+C$$

If $x=0 \rightarrow \delta=0 \rightarrow \frac{0^2}{2} = \frac{15\nu(0)}{U}+C \rightarrow C=0$

$$\delta = 5'477\sqrt{\frac{\nu x}{U}}$$

The boundary layer thickness can be expresed in adimentional form as:

$$\frac{\delta}{x} = \frac{5'477}{x}\sqrt{\frac{\nu x}{U}} \longrightarrow \frac{\delta}{x} = \frac{5'477}{\sqrt{Re}}$$

Kármán - Pohlhausen approximation (cubic)

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right)+c\left(\frac{y}{\delta}\right)^2+d\left(\frac{y}{\delta}\right)^3$$

- $u=0$ at $y=0$
- $\frac{\partial^2 u}{\partial y^2}$ at $y=0$
- $u=U$ at $y=\delta$
- $\frac{\partial u}{\partial y}=0$ at $y=\delta$

From x-momentum equation particularizated in $y=0$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{\partial^2 u}{\partial y^2}=0\,(y=0)$$

$u(0)=0$ Presure does not depend on "x"

B.C adimensionalization

- $\frac{u}{U} = 0$ at $\frac{y}{\delta} = 0 \rightarrow 0 = a + b(0) + c(0) + d(0) \rightsquigarrow \boxed{a=0}$
- $\frac{\partial^2(u/U)}{\partial(y/\delta)^2}$ at $\frac{y}{\delta} = 0 \rightarrow 0 = 2c + 6\,d(0) \longrightarrow \boxed{c=0}$
- $\frac{u}{U} = 1$ at $\frac{y}{\delta} = 1 \rightarrow 1 = b(1) + d(1)^3$
- $\frac{\partial(u/U)}{\partial(y/\delta)} = 0$ at $\frac{y}{\delta} = 1 \rightarrow 0 = b + 3d(1)$

$$\boxed{d = -1/2} \qquad \boxed{b = 3/2}$$

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Momentum integral equation (flate plate)

$$\frac{d}{dx}\int_0^{\delta}\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy = \frac{\tau_0}{\rho U^2}$$

$$\frac{d}{dx}\left[\frac{3}{2}\cdot\frac{y^2}{2\delta} - \frac{9}{4}\cdot\frac{y^3}{3\delta^2} + \frac{3}{4}\frac{y^5}{5\delta^4} - \frac{1}{2}\frac{y^4}{4\delta^3} + \frac{3}{4}\cdot\frac{y^5}{5\delta^3} - \frac{1}{4}\cdot\frac{y^7}{7\delta^6}\right]_0^{\delta} = \frac{\tau_0}{\rho U^2}$$

$$\frac{d}{dx}\left(\frac{39\delta}{280}\right) = \frac{\tau_0}{\rho U^2} \qquad \tau_0 = \mu\cdot\frac{U}{\delta}\left(\frac{3}{2}\left(1 - \cancel{\left(\frac{y}{\delta}\right)^2}^{\,0}\right)\right) = \mu\frac{U}{\delta}\cdot\frac{3}{2}$$

$$\frac{d}{dx}\left(\frac{39\delta}{280}\right) = \frac{3\mu}{2\rho U\delta} \rightsquigarrow \boxed{\frac{d\delta}{dx} = \frac{140}{13}\cdot\frac{\nu}{U\delta}} \leftarrow \text{ODE.}$$

Resolution.

$$\frac{d\delta}{dx} = \frac{140}{13}\cdot\frac{\nu}{U\delta} \Longrightarrow \delta\, d\delta = \frac{140}{13}\cdot\frac{\nu}{U}\,dx \rightarrow \int\delta\, d\delta = \int\frac{140}{13}\,\frac{\nu}{U}\,dx \rightarrow$$

$$\rightarrow \frac{\delta^2}{2} = \frac{140}{13}\cdot\frac{\nu}{U}x + \cancel{c} \rightarrow \delta^2 = \frac{280}{13}\cdot\frac{\nu}{U}x \rightarrow \delta = 4'641\sqrt{\frac{\nu x}{U}}$$

$$\delta(0) = 0 \rightarrow 0^2 = \frac{140}{13}\cdot\frac{\nu}{U}(0) + c \rightarrow c = 0$$

The boundary layer thickness is expresd in adimentional form as:

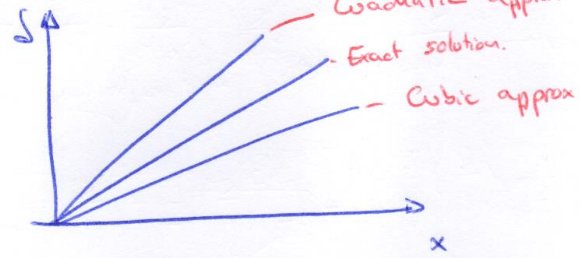
$$\frac{\delta}{x} = \frac{4'641}{x}\sqrt{\frac{\nu x}{U}} \rightsquigarrow \boxed{\frac{\delta}{x} = \frac{4'641}{\sqrt{Re}}}$$

Blasius exact solution

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re}}$$

Comparing the previous approximations to the exact solution we can observe that the cubic approximation is closer, as expected.

Plot of the solutions:



As can be seen in the schematic representation, the cubic approximation gives a thinner boundary layer regarding to the exact solution, whereas the cuadratic approximation gives a wider one.