Sònia Garrido

ADVANCED FLUID MECHANICS - HOMEWORK 2

①

a)

$$\left.\begin{array}{ll} v_r = C_1 r & r < a \\ v_r = \dfrac{C_2}{r} & r > a \end{array}\right\} V_r \rightarrow \text{velocity field}$$

Mass balance gives: $\pi a^2 V = 2\pi h v_r = Q$

$$v_r = \frac{\pi a^2 V}{2\pi h r} = \frac{a^2 V}{2hr}$$

$$\frac{a^2 V}{2hr} = \frac{C_2}{r} \; ; \quad C_2 = \frac{a^2 V}{2h}$$

$$v_r = C_1 r \Rightarrow r = a \Rightarrow v_r = C_1 a = \frac{a^2 V}{2h\cancel{a}}$$

$$C_1 = \frac{\cancel{a} V}{2h\cancel{a}} = \frac{V}{2h}$$

$$V_r = \begin{cases} \dfrac{V}{2h} r & r \leqslant a \\ \dfrac{a^2 V}{2hr} & r \geqslant a \end{cases}$$

b) Pressure field → we have to apply Bernoulli Equation along stream line from $\overrightarrow{PQR}$

$$p_0 + \frac{\rho}{2} V^2 = p(r) + \frac{\rho}{2} (v_r)^2$$

$$p(r) - p_0 = \frac{\rho}{2} V^2 - \frac{\rho}{2} (v_r)^2$$

$$p(r) - p_0 = \begin{cases} \dfrac{\rho}{2}\left(V^2 - \left(\dfrac{Vr}{2h}\right)^2\right) & r \leqslant a \\ \dfrac{\rho}{2}\left(V^2 - \left(\dfrac{a^2 V}{2hr}\right)^2\right) & r \geqslant a \end{cases}$$

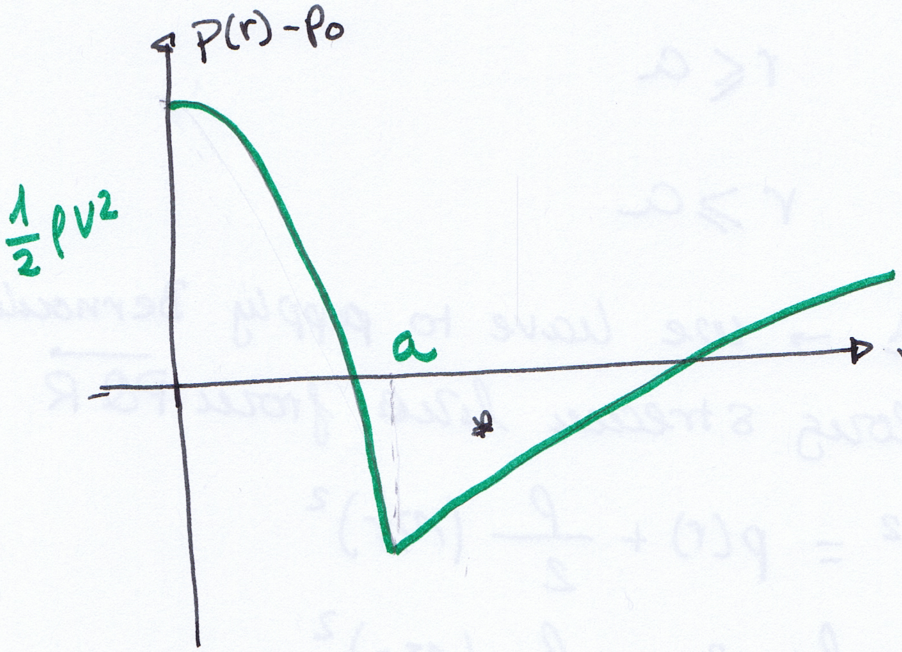
$$p(r) - p_0 = \begin{cases} \frac{\rho}{2} V^2 \left(1 - \frac{r^2}{4h^2}\right) & r \leq a \\ \frac{\rho}{2} V^2 \left(1 - \frac{a^4}{4h^2 r^2}\right) & r \geq a \end{cases}$$

$$p(r) - p_0 = 0$$

$$0 = \frac{\rho}{2} V^2 \left(1 - \frac{r^2}{4h^2}\right) \ ; \ 1 = \frac{r^2}{4h^2} \ ; \ r = 2h$$

$$p(a) - p_0 = \begin{cases} \frac{\rho}{2} V^2 \left(1 - \frac{a^2}{4h^2}\right) \\ \frac{\rho}{2} V^2 \left(1 - \frac{a^2}{4h^2}\right) \end{cases}$$

$$1 - \frac{a^2}{4h^2} \leq 0 \ ; \quad a \geq 2h \ ; \ h \leq a/2$$



* This result depends on the ~~ve~~ value of a and h, It's going to be <0 when $h \leq a/2$

c) Total Force on the disk

$$F_p = \int_0^R p(r)\, 2\pi r\, dr \ * = \int_0^a 2\pi r \frac{\rho}{2} V^2 \left(1 - \frac{r^2}{4h^2}\right) dr + \int_a^R 2\pi r \frac{\rho}{2} V^2 \left(1 - \frac{a^4}{4h^2 r^2}\right) dr_2$$

$$F_p = \pi \rho V^2 \int_0^a \left(r - \frac{r^3}{4h^2}\right) dr + \pi \rho V^2 \int_a^R \left(r - \frac{a^4 \cancel{r}}{4h^2 r^{\cancel{2}}}\right) dr =$$

$$= \pi \rho V^2 \left[\frac{r^2}{2} - \frac{r^4}{16 h^2}\right]_0^a + \pi \rho V^2 \left[\frac{r^2}{2} - \frac{a^4}{4h^2} \ln r\right]_0^R =$$

$$= \pi \rho V^2 \left[\frac{a^2}{2} - \frac{a^4}{16h^2}\right] + \pi \rho V^2 \left[\left(\frac{R^2}{2} - \frac{a^4}{4h^2} \ln R\right) - \right.$$

$$\left. - \left(\frac{a^2}{2} - \frac{a^4}{4h^2} \ln a\right)\right] = \frac{\pi \rho V^2}{2}\left(a^2 - \frac{a^4}{8h^2}\right) +$$

$$+ \pi \rho V^2 \left[\frac{R^2}{2} - \frac{a^2}{2} + \frac{a^4}{4h^2}\left(\ln(a/R)\right)\right] =$$

$$= \frac{\pi \rho V^2}{2}\left[R^2 - \frac{a^4}{8h^2} + \frac{a^4}{2h^2} \ln(a/R)\right]$$

As $a < R$ we will get $\ln(a/R) < 0$

$R^2 - \frac{a^4}{8h^2}$ can be negative when $\frac{a^4}{8h^2} \geq R^2$

$$h^2 \leq \frac{a^2}{2\sqrt{2}R}$$

$a = 1\,cm$; $R = 5\,cm$; $h = 0,1\,cm \rightarrow V?$

$w = 10\,gr$

$p(r) - p_0 = 0 \rightarrow r = 2h = 2 \cdot 0,1 = 0,2\,cm < a$

as we know that $r > a$ that means that pressure does go negative in the inner region.

$F_p + W = 0$; $F_p = -W$; $\rho_{air} = 1{,}29\,kg/m^3 = 1{,}29 \cdot 10^{-3}\,gr/c$

$$\frac{\pi \cdot 1{,}29 \cdot 10^{-3}\,V^2}{2}\left[5^2 - \frac{1}{8 \cdot 0{,}1^2} + \frac{1}{2 \cdot 0{,}1^2}\ln(1/5)\right] = -10$$

$-0{,}1377\,V^2 = -10$; $V^2 = 72{,}617$; $\boxed{V = 8{,}52\,m/s}$

d) $h = h(t)$

$$0 = \frac{DM}{Dt} = \int_{V_t} \frac{\partial \rho}{\partial t}\,dV + \int_{S_t} \rho\, v \cdot n\, dS$$

$$\left.\begin{aligned} &\frac{d}{dt}(\pi r^2 h) - \pi r^2 V + 2\pi r h\, v_r = 0 \\ &\frac{d}{dt}(\pi r^2 h) + \pi a^2 V + 2\pi r h\, v_r = 0 \end{aligned}\right\}$$

$$2\pi r h\, v_r = -\frac{d}{dt}(\pi r^2 h) + \pi r^2 V$$

$$v_r = -\frac{d}{dt}\cancel{\pi} r^{\cancel{2}} \cancel{h}\, \frac{1}{2\cancel{\pi} \cancel{r} \cancel{h}} + \frac{\cancel{\pi} r^{2} V}{2\cancel{\pi} \cancel{r} h}$$

$$\left.\begin{aligned} &v_r = \frac{rV}{2h(t)} - \frac{r}{2h(t)}\frac{dh}{dt} \qquad r \leq a \\ &v_r = \frac{a^2 V}{2 r h(t)} - \frac{r}{2h(t)}\frac{dh}{dt} \qquad r \geq a \end{aligned}\right\}$$

②

a) Boundary conditions $v(R) = 0$ → velocity on the boundary of the cylinder $= 0$.

b) $\psi(r, \theta) = f(r) \sin\theta$; $f(r) = r^{\alpha}$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

$$\alpha(\alpha-1) r^{\alpha-2} \sin\theta + \underline{\alpha r^{\alpha-2} \sin\theta} - \frac{1}{r^2} r^{\alpha} \sin\theta = 0$$

$$\alpha^2 r^{\alpha-2} \sin\theta - r^{\alpha-2} \sin\theta = 0 ; \; r^{\alpha-2} \sin\theta (\alpha^2 - 1) = 0$$

$\alpha = \pm 1$

The stream function for an uniform flow is $\psi = UR \sin\theta$. Our stream function will be a linear combination of $\alpha = 1$ and $\alpha = -1$

$$\psi = \left(Ur + \frac{B}{r}\right) \sin\theta$$

$\psi(r \to \infty) = U$ → uniform flow

$$\psi(R, \theta) = 0 = \left(UR + \frac{B}{R}\right) \sin\theta ; \; -UR^2 = B$$

$$\psi(r, \theta) = \left(Ur - \frac{UR^2}{r}\right) \sin\theta$$

⇒ velocity field

$v = -\partial\psi \ldots = \left(U \ldots UR^2\right) \sin\theta$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \left(\frac{Ur}{r} - \frac{UR^2}{r^2}\right)\cos\theta = \left(U - \frac{UR^2}{r^2}\right)\cos\theta$$

$\Rightarrow$ Pressure field

$$p^* + \frac{\rho}{2}\left(V_r^2 + V_\theta^2\right) = p^*_\infty + \frac{\rho}{2}U^2$$

$$p^* = p^*_\infty + \frac{\rho}{2}\left(U^2 - V_r^2 - V_\theta^2\right)$$

$$= p^*_\infty + \frac{\rho}{2}U^2\left(1 + \left[\left(1+\frac{R^2}{r^2}\right)^2\sin^2\theta\right] - \left[\left(1-\frac{R^2}{r}\right)^2\cos^2\theta\right.\right.$$

$\Rightarrow$ Net foces acting on the cylinder

$$\int_S \rho u v \cdot n \, dS = \int_S (-p^*)\, n \, dS$$