Homework 4. AFM

Sònia Garrido Ballart

①

a) stream function $\psi(r,\theta) = Ur^2 \sin(2\theta)$

velocity field in polar coordinates:

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{\cancel{r}} 2r^{\cancel{2}} \cos(2\theta)\cdot U$$

$$V_\theta = -\frac{\partial\psi}{\partial\theta} = -2r\sin(2\theta)\cdot U$$

The relationship between polar an cartesian coordinates is $x = r\cdot\cos\theta$ and $y = r\cdot\sin\theta$ so the velocity field in cartesian coordinates is:

$$u = 2Ux \;;\quad v = -2Uy$$

Using the Bernoulli equation we can get the pressure distribution:

$$\frac{1}{2}\left[(2Ux)^2 + (-2Uy)^2\right] + \frac{P}{\rho} - \frac{P_0}{\rho} = 0$$

$$\frac{1}{\cancel{2}}\left[\cancel{4}^{2}U^2x^2 + \cancel{4}^{2}U^2y^2\right] + \frac{P}{\rho} \cancel{+} - \frac{P_0}{\rho} = 0$$

$$\frac{P}{\rho} = \frac{P_0}{\rho} - 2U^2x^2 - 2U^2y^2$$

$$P - P_0 - 2\rho U^2(x^2+y^2)$$

b) $$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x} + \mu\nabla^2 u \\ \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) = -\dfrac{\partial p}{\partial y} + \mu\nabla^2 v \end{cases}$$

$$\begin{cases} 2U - 2U = 0 \checkmark \\ \rho(2Ux \cdot 2U - 2Uy \cdot 0) = 2\rho U^2 2x \checkmark \\ \rho(2Ux \cdot 0 - 2Uy(-2U)) = 2\rho U^2 \cdot 2y \checkmark \end{cases}$$

Potential-flow fields satisfy the eq. of motion for a viscous, incompressible fluid, but they don't satisfy the non-slip boundary condition.

c) $u = 2Ux\, f'(y)$

The continuity equation requires:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2U f'(y)$$

so that the vertical component of the velocity will be of the form:

$v = -2U f(y)$. Defining the velocity field in this way satisfies the continuity equation for all functions $f(y)$, and if we stipulate that $f(y) \to y$ as $y \to \infty$ the potential flow solution will be recovered far from the boundary.

d) y- momentum equation

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u \cdot \frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left(-2Uf(y)\left(-2Uf'(y)\right)\right) = -\frac{\partial p}{\partial y} + \mu\left(-2Uf''(y)\right)$$

$$4\rho U^2 f(y) f'(y) = -\frac{\partial p}{\partial y} - 2\mu U f''(y)$$

$$-\frac{\partial p}{\partial y} = 4\rho U^2 f(y) f'(y) + 2\mu U f''(y)$$

$$\frac{\partial p}{\partial y} = -4\rho U^2 f f' - 2\mu U f''$$

$$p(x,y) = -2\rho U^2 (f(y))^2 - 2\mu U f'(y) + g(x)$$

$g(x) \rightarrow$ Some function g x which ~~will~~ may be determined by comparison with the potential-flow pressure distribution which should be recovered for large values of y.

Recalling that $f(y) \rightarrow y$ for large values of y

$$p(x,y) \rightarrow -2\rho U^2 y^2 - 2\mu U + g(x)$$

which by comparison with the potential-flow pressure requires that:

$$g(x) = p_0 - 2\rho U^2 x^2 + 2\mu U$$

The pressure distribution in the viscous fluid will be:

$$p(x,y) = p_0 - 2\rho U^2 (f)^2 + 2\mu U(1 - f') - 2\rho U^2 x^2$$

e) x- momentum equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

$$\rho\left(2Uxf'(2Uf') - 2Uf\cdot(2Uxf''\right) = -\frac{\partial p}{\partial x} + \mu 2Uxf'''$$

$$\frac{\partial p}{\partial x} = -4\rho U^2 x$$

$$\rho\left(4U^2x(f')^2 - 4U^2xff''\right) = 4\rho U^2 x + 2U\mu x f'''$$

$$4U^2x(f')^2 - 4U^2xff'' = 4U^2x + 2U\nu x f'''$$

$$-2U\nu f''' - 4U^2xff'' + 4U^2x(f')^2 - 4U^2x = 0$$

$$\nu f''' + 2Uff'' - 2U(f')^2 + 2U = 0$$

B. C $u(x,0) = 0$ requires $f'(0) = 0$

$v(x,0) = 0$ requires $f(0) = 0$

In addition, the condition that the potential-flow solution be recovered as $y \to \infty$ requires that $f(y) \to y$ or that $f'(y) \to 1$ as $y \to \infty$

The B. C that acompany the forgoing ordinary differential equation are:

$$f(0) = f'(0) = 0$$

$$f'(y) \to 1 \text{ as } y \to \infty$$

② $$\frac{u}{U} = a + b\,\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

$u = 0$ at $y = 0$

$u = U$, $\frac{\partial u}{\partial y} = 0$ at $y = \delta$

To get the values a, b, c we have to impose the B. C

$u/U = 0$ at $y/\delta = 0$

$u/U = 1$ at $y/\delta = 1$

$$\left.\begin{aligned} &\boxed{0 = a} \\ &0 = b + 2c \cdot 1 \\ &1 = a + b + c \end{aligned}\right\} \quad \begin{aligned} &\boxed{b = 2} \\ &\boxed{c = -1} \end{aligned}$$

$$\frac{u}{U} = 2\,\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

$$\int_0^{\delta} u\,(U-u)\,dy = \int_0^{\delta} U\left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)\left(U - U\left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\right)dy =$$

$$= U^2 \int_0^{\delta} \left[2\frac{y}{\delta} - 4\frac{y^2}{\delta^2} + 2\frac{y^3}{\delta^3} - \frac{y^2}{\delta^2} + 2\frac{y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy =$$

$$= U^2 \left[\cancel{2}\frac{y^2}{\cancel{2}\delta} - 5\frac{y^3}{3\delta^2} + \cancel{4}\frac{y^4}{\cancel{4}\delta^3} - \frac{y^5}{5\delta^4}\right]_0^{\delta} =$$

$$= U^2\left[\delta - \frac{5\delta}{3} + \frac{\delta^4}{\delta^3} - \frac{\delta}{5}\right] = \frac{2\delta}{15}\,U^2$$

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_0 = \frac{2U}{\delta}\,\nu$$

substituting in the momentum integral eq.

$$\frac{\cancel{2}U^{\cancel{2}}}{15}\frac{d\delta}{dx} = \frac{\cancel{2}\cancel{U}}{\delta}\,\nu \;;\quad \frac{d\delta}{dx} = \frac{15}{U\delta}\,\nu$$

$$\delta\, d\delta = \frac{15\nu}{U}\, dx \rightarrow \int \delta\, d\delta = \int \frac{15\nu}{U}\, dx$$

$$\frac{\delta^2}{2} = \frac{15\nu x}{U} + C$$

$$\delta = 0, \quad x = 0 \longrightarrow \quad C = 0$$

$$\delta = 5,48 \sqrt{\frac{\nu x}{U}}$$

$$\boxed{\frac{\delta}{x} = \frac{5,48}{\sqrt{Re}}}$$

. Exact Blasius solution $\boxed{\frac{\delta}{x} = \frac{5}{\sqrt{Re}}}$

· Cubic velocity profile

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3$$

$$\left.\begin{array}{l} u = 0 \\ \frac{\partial^2 u}{\partial y^2} = 0 \end{array}\right\} y = 0 \qquad \left.\begin{array}{l} u = U \\ \frac{\partial u}{\partial y} = 0 \end{array}\right\} y = \delta$$

$$\left.\begin{array}{l} u/U = 0 \\ \frac{\partial^2 (u/U)}{\partial (y/\delta)^2} = 0 \end{array}\right| y/\delta = 0 \qquad \left.\begin{array}{l} u/U = 1 \\ \frac{\partial (u/U)}{\partial (y/\delta)} = 0 \end{array}\right\} y/\delta = 1$$

$a = 0$

$b = \frac{3}{2}$

$c = 0$

$d = -\frac{1}{2}$

→ Imposing the B.C we obtain the values of a, b, c, d.

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\int_0^\delta U\left(\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{8}\right)^3\right)\left(U - U\left(\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{8}\right)^3\right)\right)dy$$

$$= U^2\int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{9}{4}\frac{y^2}{\delta^2} + \frac{3}{2}\frac{y^4}{\delta^4} - \frac{1}{2}\frac{y^3}{\delta^3} - \frac{1}{4}\frac{y^6}{\delta^6}\right]dy =$$

$$= U^2\left[\frac{3}{2}\frac{y^2}{2\delta} - \frac{\cancel{9}3}{4}\frac{y^3}{\cancel{3}\delta^2} + \frac{3}{2}\frac{y^5}{5\delta^4} - \frac{1}{2}\frac{y^4}{4\delta^3} - \frac{1}{4}\frac{\delta^7}{7\delta^6}\right]_0^\delta =$$

$$= \frac{48\delta U^2}{280}$$

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_0 = \frac{3}{2\delta}U\nu$$

Substituting in the momentum integral eq

$$\frac{3}{2\delta}U\nu = \frac{48\delta U^2}{280}\frac{d\delta}{dx}$$

$$\delta\, d\delta = \frac{840}{96U}\nu\, dx = \frac{35}{4U}\nu\, dx$$

$$\int \delta \, d\delta = \int \frac{35\nu}{4U} dx \; ; \quad \frac{\delta^2}{2} = \frac{35\nu x}{4U} + C$$

$$\delta = 0, \quad x = 0 \longrightarrow \quad C = 0$$

$$\delta^2 = \frac{35\nu x}{2U} \; ; \quad \delta = 4{,}18\sqrt{\frac{\nu x}{U}}$$

$$\boxed{\frac{\delta}{x} = \frac{4{,}18}{\sqrt{Re}}}$$