HomeWork 4

Advanced Fluid Mechanics

Parris Dilip Mulye

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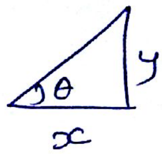
Problem 1

(a)

$$\psi(r,\theta) = U r^2 \sin 2\theta$$



$$r = x^2+y^2 \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{2y}{\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{2xy}{x^2+y^2}$$

$$\psi(x,y) = \frac{U(x^2+y^2)\,2xy}{(x^2+y^2)} = 2xyU$$

$$\psi(x,y) = 2xyU$$

$$u = \frac{\partial\psi}{\partial y} = 2xU \qquad v = -\frac{\partial\psi}{\partial x} = -2yU$$

Boundary conditions,

1. at $x=0$ (along Y axis) $u=0$ (Symmetry + flow velocity)

$$u\big|_{x=0} = 2\times 0\times U = 0 \quad \text{satisfied.}$$

2. at $y=0$ (along X axis) $V=0$ (cannot penetrate or move away the boundary)

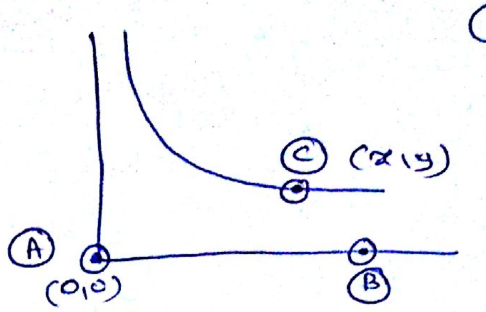
$$v\big|_{y=0} = -2\times 0\times U = 0 \quad \text{satisfied.}$$

Note: No slip boundary cannot be applied, since this is assumed to be ideal fluid (It can slip on the ~~solid~~ boundaries)

~~Applying Bernoullis theorem~~

$u = 2xU$

$v = -2yU$



$\nabla \times \overline{V} = 0$ Irrotational flow,

so Bernoulis theorem can be applied between any two points in space.

Between (A) & (C)

$$P_0 + 0 = P_x + \frac{1}{2}\rho |\overline{V}|^2$$

$\cancel{\Rightarrow P_x}$.

$$P_x = P_0 - \frac{1}{2}\rho\left[(2xU)^2 + (-2yU)^2\right]$$

$$\boxed{P = P_0 - 2\rho U^2(x^2+y^2)}$$

(b) Navier-stokes for 2D flows & no body forces

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (X)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (Y)$$

(X)

LHS RHS

$$\rho\left(0 + (2xU)(2U) + 0\right) = -\frac{\partial p}{\partial x} + \mu(0+0)$$

LHS RHS

$$4U^2x\rho = +2\rho U^2(2x)$$

from pressure distribution obtained in (a)

LHS RHS

$$4U^2x\rho \qquad 4\rho U^2x$$

LHS = RHS

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(Y)

LHS RHS

$$\rho\left[0 + 0 + (-2yU)(-2U)\right] \cdot \qquad \frac{-\partial p}{\partial y} + \mu(0+0)$$

$$+ 4yU^2\rho \qquad + (2)\rho U^2(2y)$$

LHS = RHS.

Hence, the former velocity & pressure satisfy the Navier-Stokes equations.

BC for the viscous problem, (in addition to previous BCs)

NO-slip BC on $(y=0)$

so $$u\big|_{y=0} = 0$$

But $u\big|_{y=0} = 2xU \neq 0$ so not satisfied.

(c) $u = 2Uxf'$ f is a function only of y

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -2Uf'$$

$$\boxed{v = -2Uf}$$

Based on BCs.

① No normal velocity at boundary (stationary)

$v\big|_{y=0} = 0 \Rightarrow f(0) = 0$

② No slip at boundary

$u\big|_{y=0} = 0 \Rightarrow f'(0) = 0$

③ At large Y (far away from Boundary) old solution

$u\big|_{y\to\infty} = 2xU \Rightarrow f'(\infty) = 1$ $v\big|_{y\to\infty} = -2uf \Rightarrow f(\infty) = y$

Both are same

(d) Y-momentum with $u = 2xUf'$ $\quad v = -2Uf$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho(2xUf')(0) + \rho(-2Uf)(-2Uf') = -\frac{\partial p}{\partial y} + \mu\left(0 + (-2Uf'')\right)$$

$$4U^2 ff'\rho = -\frac{\partial p}{\partial y} - 2U\mu f''$$

$$\frac{\partial p}{\partial y} = -4U^2 ff'\rho + 2U\mu f''$$

RHS = only function of y so integrating directly.

$$p = -2U^2\rho f^2 - 2U\mu f' + m(x)$$

$$p(x,y) = -2U^2\rho f^2 - 2U\mu f' + m(x)$$

Recovery of old flow at $y \to \infty$

so p should approach old pressure value.

$$P_0 - 2\rho U^2(x^2+y^2) = -2U^2\rho y^2 - 2U\mu + m(x)$$

(since $f(\infty) = y$ & $f'(\infty) = 1$)

$$m(x) = P_0 - 2\rho U^2 x^2 + 2U\mu$$

$$\therefore \boxed{p(x,y) = P_0 - 2\rho U^2 f^2 + 2\mu U(1-f') - 2\rho U^2 x^2}$$

(e) X-momentum.

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu(0 + 2Uxf''')$$

$$\rho(2xUf')(2Uf') - \rho(2Uf)(2Uxf'') = -\frac{\partial p}{\partial x} + \mu 2Uxf'''$$

$$\rho 4xU^2(f')^2 - 4U^2x\rho ff'' = -2\rho U^2(2x) + 2\mu Uxf'''$$

↳ from pressure calculated in (d)

Cancelling x, and dividing by $4\rho U^2$ on all sides

$$(f')^2 - ff'' = 1 + \frac{\nu}{2U}f''' \qquad \left(\nu = \frac{\mu}{\rho}\right)$$

$$\boxed{\frac{\nu}{2U}f''' + ff'' - (f')^2 + 1 = 0}$$

BCs. from (c) part.

1. $f(\infty) = y$
2. $f'(0) = 0$
3. $f(0) = 0$

So this is a third order ODE in f with 3 BCs but it is non-linear. Can be solved numerically with RK method or any such method.

Problem 2

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2 \quad \text{(given)}$$

Boundary conditions given are

① $u = 0$ at $y = 0$

$$\Rightarrow \quad 0 = a + 0 + 0$$

$$\boxed{a = 0}$$

② $u = U$ at $y = \delta$

$$\Rightarrow \quad 1 = 0 + b + c$$

$$\boxed{b + c = 1}$$

③ $\frac{\partial u}{\partial y} = 0$ at $y = \delta$

$$\frac{1}{U}\frac{\partial u}{\partial y} = \frac{b}{\delta} + \frac{c}{\delta^2} 2y$$

at $y = \delta$

$$0 = \frac{b}{\delta} + \frac{2c}{\delta}$$

$$\boxed{b + 2c = 0}$$

$$\Rightarrow \boxed{b = 2 \qquad c = -1}$$

$$\boxed{u = U\frac{2y}{\delta} - \frac{Uy^2}{\delta^2}}$$

Boundary Layer Equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

for flat plate, these equations when integrated and combined,

$$\frac{d}{dx}\int_0^\infty u(U-u)\,dy=\frac{\tau_0}{\rho}$$

$$\int_0^\infty=\int_0^\delta+\int_\delta^\infty \quad \hookrightarrow u=U \Rightarrow (U-u)=0$$

$$=\int_0^\delta$$

$$\int_0^\delta u(U-u)\,dy=U^2\int_0^\delta\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)dy$$

$$\frac{y}{\delta}=p$$

$$dy=dp\,\delta$$

$$=\delta U^2\int_0^1(2p-p^2)(1-2p+p^2)\,dp$$

$$=\delta U^2\int_0^1\left(2p-4p^2+2p^3-p^2+2p^3-p^4\right)dp$$

$$=\delta U^2\left(\frac{2p^2}{2}-\frac{4p^3}{3}+\frac{2p^4}{4}-\frac{p^3}{3}+\frac{2p^4}{4}-\frac{p^5}{5}\right)_0^1$$

$$=\delta U^2\left(1-\frac{4}{3}+\frac{1}{2}-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}\right)$$

$$=\delta U^2\left(2-\frac{1}{5}-\frac{5}{3}\right)=\delta U^2\left(\frac{2}{15}\right)$$

$$\text{LHS}=\frac{d}{dx}\left(\frac{\delta(U^2)2}{15}\right)=\boxed{\frac{2U^2}{15}\frac{d\delta}{dx}}$$

RHS -

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)\Bigg|_{y=0}$$

$$= \nu U\left(\frac{2}{\delta} - \frac{2yU}{\delta^2}\right)\Bigg|_{y=0}$$

$$= \frac{2U\nu}{\delta}$$

LHS = RHS

$$\frac{\cancel{2}U^{\cancel{2}}}{15}\frac{d\delta}{dx} = \frac{\cancel{2}\cancel{U}\nu}{\delta}$$

$$\int \delta\, d\delta = \int 15\left(\frac{\nu}{U}\right) dx$$

$$\frac{\delta^2}{2} = 15\frac{\nu}{U}x$$

$$\delta = \sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\boxed{\delta = 5.4772\sqrt{\frac{\nu x}{U}}} \Rightarrow \boxed{\frac{\delta}{x} = \frac{5.4772}{\sqrt{Re}}}$$

Blasius $\Rightarrow \delta = 5\sqrt{\frac{\nu x}{U}}$ cubic $\Rightarrow$ ~~4.64~~ $\delta = 4.64\sqrt{\frac{\nu x}{U}}$

So δ for quadratic assumption is the highest among Blasius & Cubic assumption.

Momentum thickness (θ)

$$\cancel{\frac{\tau_0}{\frac{1}{2}\rho U^2}} \cancel{=} \cancel{\frac{2U\nu}{\delta\frac{1}{2}U^2}} = \quad\quad U^2\theta = \int_0^\infty u(U-u)dy$$

$$U^2\theta = \frac{2U^2\delta}{15} \quad\quad \theta = \frac{2\delta}{15} = \frac{2}{15}\sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\frac{\theta}{x} = \frac{2\sqrt{30}}{15}\frac{1}{\sqrt{Re}} = \boxed{\frac{0.7303}{\sqrt{Re}}}$$

Blasius $\Rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}}$ cubic $\Rightarrow \frac{\theta}{x} = \frac{0.646}{\sqrt{Re}}$

As expected from δ, $\frac{\theta}{x}$ for quadratic is more than both Blasius & cubic.