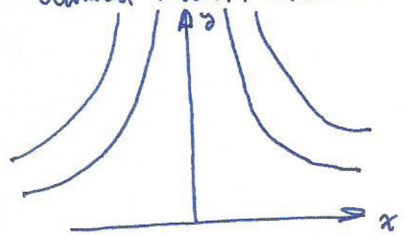
Homework 4 Martí Bonet Rodríguez

Advanced Fluid Mechanics



(1)

a) Consider ideal fluid. Compute velocity in cartesian coordinates (u, v) from the stream function:

$\psi(r,\theta) = Vr^2 \sin(2\theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\psi(r,\theta) = Vr^2\sin(2\theta) = 2Vr\sin\theta \, r\cos\theta = 2Vxy = \psi(x,y)$ Stream Function in Cartesian Coordinates

$$\begin{cases} u = \dfrac{\partial\psi}{\partial y} = 2Vx \\ v = -\dfrac{\partial\psi}{\partial x} = -2Vy \end{cases}$$ Velocities in Cartesian Coordinates

- As the flow is symmetric, in the y-axis $(x=0)$ the horizontal velocity has to be zero. Likewise at $y=0$, the velocity in the vertical direction has to be 0. Therefore $(0,0)$ is a Stagnation Point:

$u(0) = 2V(0) = 0$

$v(0) = -2V(0) = 0$

Stagnation Point

- The pressure can be computed using Bernoulli's Equation from the Stagnation Point:

$$\int_1^2 \frac{\partial v}{\partial t}\cdot ds + \left[\frac{1}{2}v_2^2 + \frac{P_2}{\rho} - f_2\right] - \left[\frac{1}{2}v_1^2 + \frac{P_1}{\rho} - f_1\right] = 0 \quad ; \; v_2^2 = (u^2+v^2)$$

Stationary No body forces Stagnation No body forces

$$P_2 = P_1 - \frac{1}{2}\rho v_2^2 = P_1 - \frac{1}{2}\rho\left[(2Vx)^2 + (-2Vy)^2\right] = P_1 - \frac{1}{2}\rho\left[4V^2x^2 + 4V^2y^2\right] =$$

Pressure Distribution: $\boxed{P_2 = P_1 - 2V^2\rho(x^2+y^2)}$

b) Show that velocity and pressure satisfies Navier-Stokes equations, but not B.C.'s for viscous problem:

X Momentum:

$$\rho\left[\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Steady}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \underbrace{\cancel{\rho b_x}}_{\text{No body forces}}$$

$$\rho\left[2Vx(2V) - 2Vy(0)\right] = +4\rho V^2 x + \mu(0+0) \Longrightarrow 4\rho V^2 x = 4\rho V^2 x \quad ✓$$

Y Momentum:

$$\rho\left[\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\text{Steady}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \underbrace{\cancel{\rho b_y}}_{\text{No body forces}}$$

$$\rho\left[2Vx(0) - 2Vy(-2V)\right] = +4\rho V^2 y + \mu(0+0) \Longrightarrow 4\rho V^2 y = 4\rho V^2 y \quad ✓$$

- For the viscous problem there should be non-slip B.C. on $y=0$:

$$u(x,0) = 2Vx \neq 0$$

c) Show that boundary conditions are met for:

$$u = 2Vx f'(y)$$

State boundary conditions for f:

- From Continuity Equation for an incompressible flow: $\nabla \cdot v = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2Vf'(y) + \frac{\partial v}{\partial y} = 0 \longrightarrow \frac{\partial v}{\partial y} = -2Vf'(y)$$

- Integrating:

$$v = -\int 2Vf'(y)\,dy = -2Vf(y)$$

B.C.: far from the boundary, the viscous effect is negligible so it should be equal to potential flow.

$$v(y\to\infty) = -2Vf(y) = -2Vy \longrightarrow f(y) = \frac{2V}{2V}y = y$$

B.C: at $y=0$ there is a stagnation point, so $v(0)=0$.

$$v(0) = -2Vf(0) = 0 \longrightarrow f(0) = 0$$

d) Use y- momentum equation to obtain an expression for pressure distribution.

$$\rho\left[\cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \cancel{\rho b_y}$$

Stationary

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\nabla^2 v$$

$$\rho\left[2Vx f'(y)(0) - 2Vf(y)(-2Vf'(y))\right] = \frac{-\partial p}{\partial y} + \mu\left[0 - 2Vf''(y)\right]$$

$$\frac{\partial p}{\partial y} = -4V^2\rho f(y)f'(y) - 2V\mu f''(y)$$

$$* \; 2f(y)f'(y) = f'(y)^2$$

· Integrating :

$$p = -\int 4V^2\rho f(y)f'(y) - 2V\mu f''(y)\,dy = -2V^2\rho f(y)^2 - 2V\mu f'(y) + h(x)$$

· For $y \to \infty$, potential flow pressure must be recovered : $P_2 = P_1 - 2V^2\rho(x^2+y^2)$

· From B.C. we know that $f(y) = y$

$$P = -2V^2\rho \cancel{y^2} - 2V\mu + h(x) = P_1 - 2V^2\rho(x^2 + \cancel{y^2})$$

$$h(x) = P_1 - 2V^2\rho x^2 + 2V\mu = P_1 + 2V^2(\mu - \rho x^2)$$

e) Use x- momentum equation and pressure distribution to obtain a differential equation for f.

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\rho\left[2Vx f'(y)\,2Vf'(y) - 2Vf(y)\,2Vxf''(y)\right] = -\frac{\partial p}{\partial x} + \mu\left[0 + 2Vxf'''(y)\right]$$

$$4V^2\rho x\left(f'(y)^2 - f(y)f''(y)\right) = -\frac{\partial p}{\partial x} + 2V\mu x f'''(y)$$

$$\frac{\partial p}{\partial x} = -4V^2 x$$

$$4V^2\rho x\left(f'(y)^2 - f(y)f''(y)\right) = 4V^2 x + 2V\mu x f'''(y) \longrightarrow \frac{\mu}{2V}f'''(y) + \rho f(y)f''(y) - \rho f'(y)^2 + 1 = 0$$

$$\nu = \frac{\mu}{\rho}$$

Differential Equation

$$\boxed{\frac{\nu}{2V}f'''(y) + f(y)f''(y) - f'(y)^2 + 1 = 0}$$

$\longrightarrow$

B.C.'s:

$$\left.\begin{array}{l} f(y\to\infty) = y \\ f(y=0) = 0 \end{array}\right\} \text{From c)}$$

$u(x,0) = 2Vx f'(0) = 0 \longrightarrow f'(0) = 0$ Non-slip B.C.

$u(x, y\to\infty) = 2Vx f'(y) = 2Vx \longrightarrow f'(y\to\infty) = 1$ Potential Flow

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Homework 4

Advanced Fluid Mechanics

a) Use the Kármán - Pohlhausen approximation to compute the boundary layer. Assume the quadratic polynomial velocity:

$$\frac{u}{V} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \qquad \text{with:} \quad \begin{array}{l} u=0 \quad \text{at } y=0 \\ u=V,\ \dfrac{\partial u}{\partial y}=0 \quad \text{at } y=\delta \end{array}$$

To determine the coefficients we apply B.C.'s:

$$u = V\left(a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2\right) \qquad \frac{\partial u}{\partial y} = \frac{b}{\delta} + 2c\frac{y}{\delta^2}$$

$$u(0) = Va = 0 \longrightarrow \boxed{a=0}$$

$$\frac{\partial u(\delta)}{\partial y} = \left[\frac{b}{\delta} + 2c\frac{\cancel{\delta}}{\cancel{\delta^2}}\right]V = \left[\frac{b}{\delta} + \frac{2c}{\delta}\right]V = \left[\frac{1-c+2c}{\delta}\right]\cancel{V} = 0$$

$$u(\delta) = V\left[b\frac{\cancel{\delta}}{\cancel{\delta}} + c\left(\frac{\cancel{\delta}}{\cancel{\delta}}\right)^2\right] = V(b+c) = V \to b = 1-c \qquad \boxed{b=2}$$

$$\frac{1+c}{\delta} = 0 \to \boxed{c=-1}$$

$$\frac{u}{V} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

The momentum thickness equation says:

$$\frac{d}{dx}(V^2\theta) + \delta V \cancel{\frac{dV}{dx}} = \frac{\tau_0}{\rho} \longrightarrow \frac{d}{dx}(V^2\theta) = \frac{\tau_0}{\rho}$$

only depends on y

The integral form of the above equation is:

$$\frac{d}{dx}(V^2\theta) = \frac{d}{dx}\int_0^\infty u(V-u)\,dy = \frac{\tau_0}{\rho}$$

Substituting the expression of u obtained before and integrating between 0 and δ:

$$\int_0^\delta u(V-u)\,dy = \int_0^\delta V\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\left(V - \left[V\left(2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right)\right]\right)dy = \int_0^\delta V^2\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] + \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]^2 dy$$

$$= V^2\left[\left(\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right) - \left(\frac{4}{3}\frac{y^3}{\delta^2} + \frac{y^5}{5\delta^4} - \frac{4}{4}\frac{y^4}{\delta^3}\right)\right]_0^\delta = V^2\left[\cancel{\delta} - \frac{\delta}{3} - \frac{4}{3}\delta + \frac{\delta}{5} - \cancel{\delta}\right] = \frac{V^2\delta^2}{2}$$

The shear stress is computed as:

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_0 = \nu\left[\frac{2V}{\delta}\right] = \frac{2\nu V}{\delta}$$

$$\frac{d}{dx}\left(\frac{V^2\delta^2}{2}\right) = \frac{2V^2}{2}\frac{d\delta}{dx} = V^2\frac{d\delta}{dx} = \frac{2\nu V}{\delta} \longrightarrow \delta^2 = \frac{2\nu}{V}x \longrightarrow \left[\delta = 1'4142\sqrt{\frac{\nu x}{V}}\right]$$

$$\frac{\tau_0}{\frac{1}{2}\rho V^2} = \frac{2\nu V}{\frac{1}{2}\delta V^2} = \frac{4\nu}{\delta V} = \frac{4}{\sqrt{Re}}$$

(Displacement thickness)

The Blasius solution gives:

$$\frac{\theta}{x} = \frac{0'664}{\sqrt{Re}} \quad \text{and} \quad \frac{\delta}{x} = \frac{5}{\sqrt{Re}}$$

(Momentum thickness) (Displacement thickness)

Assuming cubic profile, the Kármán - Pohlhausen solution results in:

$$\delta = 4'64\sqrt{\frac{\nu x}{V}} \quad \text{and} \quad \frac{\tau_0}{\frac{1}{2}\rho V^2} = \frac{0'646}{\sqrt{Re}}$$

$$\delta = 4'64 \cdot \frac{1}{\sqrt{Re}}$$

We can see that the assumption of cuadratic velocity profile gives reasonable good results of the boundary layer because in the Blasius solution $\frac{\delta}{x}$ is $\frac{5}{\sqrt{Re}}$ and in the cubic profile $\frac{4'64}{\sqrt{Re}}$ while in our case is $\frac{4}{\sqrt{Re}}$. We can be confident with this result because is consistent with the other results.