Advanced Fluid Mechanics

Homework 4

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1) $\psi(r,\theta) = Ur^2 \sin 2\theta$

$r = x^2+y^2$ $\qquad$ $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{2y}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$$

$$\psi(x,y) = \frac{U(x^2+y^2)\,2xy}{x^2+y^2} = 2xyU$$

$$\psi(x,y) = 2xyU$$

$$u = \frac{\partial \psi}{\partial y} = 2xU \quad ; \quad v = -\frac{\partial \psi}{\partial x} = -2yU$$

Boundary Conditions

1. at $x=0$, $u=0$ (symmetry + flow velocity)

   $u|_{x=0} = 2\times 0\times U = 0$ satisfied

2. at $y=0$, $v=0$

   $v|_{y=0} = -2\times 0\times U = 0$ satisfied

No slip boundary cant be applied as this is assumed to be ideal fluid, so it can slip on the boundaries

$u = 2xU$

$v = -2yU$

$\nabla\times\bar{V} = 0$ Irrotational flow

So bernoulli's theorem can be applied between any two points.

Applying bernoulli's equation

$$P_0 + 0 = P_x + \frac{1}{2}\rho|\bar{v}|^2$$

$$\Rightarrow P_x = P_0 + \frac{1}{2}\rho\left[(2xU)^2 + (-2yU)^2\right]$$

$$\boxed{P = P_0 - 2\rho U^2(x^2+y^2)}$$

b) Navier - Stokes for 2D flows & no body forces

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad y$$

$\underline{x}$ $\rho(0 + (2xU)(2U) + 0) - \frac{\partial P}{\partial x} + \mu(0+0)$

LHS $= 4U^2 x\rho$, RHS $= 2\rho U^2(2x)$

$= 4\rho U^2 x$

$\therefore$ LHS = RHS

$\underline{y}$

LHS $= \rho\left[0 + 0 + (-2yU)(-2U)\right]$

$= 4yU^2\rho$

RHS $= -\frac{\partial p}{\partial y} + \mu(0)$

$= 4\rho y U^2$

LHS = RHS

Hence the former velocity & pressure satisfy the Navier Stokes equations

BC for the viscous problem, (in addition to previous BC(s))

No - slip BC on $y=0$

$$u|_{y=0} = 0$$

But $u|_{y=0} = 2xU \neq 0$ so not satisfied

c) $u = 2Uxf'$ where f is a function of y

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -2Uf'$$

$$\boxed{v = -2Uf}$$

Based on BCs,

① No normal velocity at boundary

$$v|_{y=0} = 0 \Rightarrow f(0) = 0$$

② No slip at boundary

$$u|_{y=0} = 0 \Rightarrow f'(0) = 0$$

③ At large y (far away from boundary)

$$u|_{y\to\infty} = 2xU \Rightarrow f'(\infty) = 1 \quad . \quad v|_{y\to\infty} = -2uf \Rightarrow f(\infty) = y$$

Both are same

d) Y-momentum with $u = 2xUf'$, $v = -2Uf$

$$\rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\Rightarrow \rho(2xUf')(0) + \rho(-2Uf)(-2Uf') = -\frac{\partial p}{\partial y} + \mu(0 + (-2uf''))$$

$$\Rightarrow 4U^2ff'\rho = -\frac{\partial p}{\partial y} = -2U\mu f''$$

$$\Rightarrow \frac{\partial p}{\partial y} = -4U^2ff'\rho + 2U\mu f''$$

RHS is only function of y & integrating directly

$$p = -2U^2\rho f^2 - 2U\mu f' + m(x)$$

Recovery of old flow at $y \to \infty$

so p should approach old pressure value

i.e. $P_0 - 2\rho U^2 (x^2 + y^2) = -2U^2 \rho y^2 - 2U\mu + m(x)$

(Since $f(\infty) = y$ & $f'(\infty) = 1$)

$m(x) = P_0 - 2\rho U^2 x^2 + 2U\mu$

$$\therefore \boxed{p(x,y) = P_0 - 2\rho U^2 f^2 + 2\mu U(1-f') - 2\rho U^2 x^2}$$

e) X - momentum

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu(0 + 2Ux f''')$$

$$\Rightarrow \rho\cdot(2xUf')(2Uf') - \rho(2Uf)(2Uxf'') = -\frac{\partial P}{\partial x} + 2\mu U x f'''$$

$$\Rightarrow 4\rho x U^2 (f')^2 - 4U^2 x \rho f f'' = -2\rho U^2(2x) + 2\mu U x f'''$$

Cancelling x, and dividing by $4\rho U^2$ on all sides

$$(f')^2 - ff'' = 1 + \frac{\nu}{2U} f''' \quad \left(\text{where } \nu = \frac{\mu}{\rho}\right)$$

$$\boxed{\frac{\nu}{2U} f''' + ff'' - (f')^2 + 1 = 0}$$

BCs from (c) part

① $f(\infty) = y$

② $f'(0) = 0$

③ $f(0) = 0$

This is a third order ODE in f with 3 BCs and it is non-linear. It can be solved numerically with Range-Kutta method or any such method.

2) $\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$

BCs given

① $u = 0$ at $y = 0$

$\Rightarrow 0 = a + 0 + 0$

$\Rightarrow a = 0$

② $u = U$ at $y = \delta$

$\Rightarrow 1 = 0 + b + c$

$\Rightarrow b + c = 1$

③ $\frac{\partial u}{\partial y} = 0$ at $y = \delta$

$\frac{1}{U}\frac{\partial u}{\partial y} = \frac{b}{\delta} + \frac{c}{\delta^2} 2y$

at $y = \delta$

$\frac{b}{\delta} + \frac{2c}{\delta} = 0$

$b + 2c = 0$

$\therefore \boxed{b = 2 \quad , c = -1}$

$$\boxed{u = \frac{U 2y}{\delta} - \frac{U y^2}{\delta^2}}$$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$

for flat plate, these equations when integrated and combined

$$\frac{d}{dx}\int_0^\infty u(U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\Rightarrow \int_0^\delta u(U-u)\,dy = U^2\int_0^\delta\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right)dy$$

$$\boxed{\begin{aligned} \frac{y}{\delta} &= p \\ dy &= \delta\, dp \end{aligned}}$$

$$\Rightarrow \delta U^2 \int_0^1 (2p - p^2)(1 - 2p + p^2)\, dp$$

$$= \delta U^2 \int_0^1 (2p - 4p^2 + 2p^3 - p^2 + 2p^3 - p^4)\, dp$$

$$= \delta U^2 \left[\frac{2p^2}{2} - \frac{4p^3}{3} + \frac{2p^4}{4} - \frac{p^3}{3} + \frac{2p^4}{4} - \frac{p^5}{5} \right]_0^1$$

$$= \delta U^2 \left(1 - \frac{4}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{5}\right)$$

$$= \frac{2}{15} \delta U^2$$

$$\underline{LHS} = \frac{d}{dx}\left(\frac{2}{15} \delta U^2\right) = \boxed{\frac{2U^2}{15} \frac{d\delta}{dx}}$$

$$\underline{RHS},\ \frac{\tau_0}{\rho} = \nu \left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} = \nu U \left(\frac{2}{\delta} - \frac{2Uy}{\delta^2}\right)\bigg|_{y=0} = \frac{2U\nu}{\delta}$$

LHS = RHS

$$\Rightarrow \frac{\cancel{2}U^{\cancel{2}}}{15}\frac{d\delta}{dx} = \frac{\cancel{2U}\nu}{\delta} \Rightarrow \int \delta\, d\delta = \int 15\left(\frac{\nu}{U}\right) dx$$

$$\Rightarrow \frac{\delta^2}{2} = 15 \frac{\nu}{U} x$$

$$\delta = \sqrt{30 \frac{\nu x}{U}} \Rightarrow \delta = 5.4772 \sqrt{\frac{\nu x}{U}}$$

$$\Rightarrow \frac{\delta}{x} = \frac{5.4772}{\sqrt{Re}}$$

Blasius $\Rightarrow \delta = 5\sqrt{\frac{\nu x}{U}}$ cubic $\Rightarrow \delta = 4.64\sqrt{\frac{\nu x}{U}}$

So δ for quadratic assumption is the highest among Blasius & cubic assumption

Momentum thickness(θ), $U^2\theta = \int_0^\infty u(U-u)\, dy$

$$\Rightarrow U^2\theta = \frac{2U^2\delta}{15} \Rightarrow \theta = \frac{2\delta}{15} = \frac{2}{15}\sqrt{\frac{30\nu x}{U}} \Rightarrow \frac{\theta}{x} = \frac{2\sqrt{30}}{15}\frac{1}{\sqrt{Re}} = \boxed{\frac{0.7303}{\sqrt{Re}}}$$

Blasius $\Rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}}$, cubic $\Rightarrow \frac{\theta}{x} = \frac{0.646}{\sqrt{Re}}$

As expected from $\frac{\delta}{x}$, $\frac{\theta}{x}$ for quadratic is more than both Blasius & cubic