Advanced Fluid Dynamics

30/10/2015
Albert Gancedo

$\boxed{1(a)}$

$$\nabla \cdot (\nabla \times F) = 0$$

$$\nabla \cdot (\nabla \times F) = \nabla \cdot \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) e_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) e_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) e_3$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) \overset{1}{\cancel{e_1 e_1}} + \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) \overset{1}{\cancel{e_2 e_2}} + \frac{\partial}{\partial x_3}\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \overset{1}{\cancel{e_3 e_3}}$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_3}{\partial x_2} + \frac{\partial F_2}{\partial x_3} - \frac{\partial F_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_1}{\partial x_3}\right) = 0$$

EXAMPLE

$$\boxed{\frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} F_1 = \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} F_1}$$

(b) $\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F$

$$\nabla \times (\nabla \times F) = \left[\frac{\partial}{\partial x_2} \frac{\partial F_2}{\partial x_1} - \frac{\partial^2 F_1}{\partial x_2}\right] e_2 - \left[\frac{\partial^2 F_1}{\partial x_3} - \frac{\partial}{\partial x_3} \frac{\partial F_3}{\partial x_1}\right] e_3 + \left[\frac{\partial}{\partial x_3} \frac{\partial F_3}{\partial x_2} - \frac{\partial^2 F_2}{\partial x_3}\right] e_3$$

$$- \left[\frac{\partial^2 F_2}{\partial x_1} - \frac{\partial}{\partial x_1} \frac{\partial F_1}{\partial x_2}\right] e_1 + \left[\frac{\partial}{\partial x_1} \frac{\partial F_1}{\partial x_3} - \frac{\partial^2 F_3}{\partial x_1}\right] e_1 - \left[\frac{\partial^2 F_3}{\partial x_2} - \frac{\partial}{\partial x_2} \frac{\partial F_2}{\partial x_3}\right]$$

$$= \left[\frac{-\partial^2}{\partial x_1} (F_2 + F_3) + \frac{\partial}{\partial x_1}\left(\frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\right)\right] e_1 + \left[\frac{-\partial^2}{\partial x_2} (F_1 + F_3) + \frac{\partial}{\partial x_2}\left(\frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial x_3}\right)\right] e_2$$

$$+ \left[\frac{-\partial^2}{\partial x_3} (F_1 + F_2) + \frac{\partial}{\partial x_3}\left(\frac{\partial F_3}{\partial x_1} + \frac{\partial F_3}{\partial x_2}\right)\right] e_3$$

$$\nabla(\nabla\cdot F)-\nabla^2F=\left[\cancel{\frac{\partial^2F_1}{\partial x_1}}+\frac{\partial}{\partial x_2}\frac{\partial F_1}{\partial x_1}+\frac{\partial\,\partial F_1}{\partial x_3\,\partial x_1}\right]e_1$$

$$+\left[\cancel{\frac{\partial^2F_2}{\partial x_2}}+\frac{\partial\,\partial F_2}{\partial x_1\,\partial x_2}+\frac{\partial\,\partial F_2}{\partial x_3\,\partial x_2}\right]e_2+\left[\frac{\partial\partial F_3}{\partial x_1\,\partial x_3}+\frac{\partial\,\partial F_3}{\partial x_2\,\partial x_3}+\cancel{\frac{\partial^2F_3}{\partial x_3}}\right]e_3$$

$$-\left[\cancel{\frac{\partial^2F_1}{\partial x_1}}+\frac{\partial^2F_2}{\partial x_1}+\frac{\partial^2F_3}{\partial x_1}\right]e_1-\left[\frac{\partial^2F_1}{\partial x_2}+\cancel{\frac{\partial^2F_2}{\partial x_2}}+\frac{\partial^2F_3}{\partial x_2}\right]e_2-\left[\frac{\partial^2F_1}{\partial x_3}+\frac{\partial^2F_2}{\partial x_3}+\cancel{\frac{\partial^2F_3}{\partial x_3}}\right]e_3$$

$$=\left[\frac{-\partial^2}{\partial x_1}(F_2+F_3)+\frac{\partial}{\partial x_1}\left(\frac{\partial F_1}{\partial x_2}\ \frac{\partial F_1}{\partial x_3}\right)\right]e_1+\left[\frac{-\partial^2}{\partial x_2}(F_1+F_3)+\frac{\partial}{\partial x_2}\left(\frac{\partial F_2}{\partial x_1}+\frac{\partial F_2}{\partial x_3}\right)\right]e_2$$

$$+\left[\frac{-\partial^2}{\partial x_3}(F_1+F_2)+\frac{\partial}{\partial x_3}\left(\frac{\partial F_3}{\partial x_1}+\frac{\partial F_3}{\partial x_2}\right)\right]=\nabla\times(\nabla\times F)$$

(c) $\nabla\cdot(F\times G)=G\cdot\nabla\times F-F\cdot\nabla\times G$

$$\nabla\cdot(F\times G)=\frac{\partial(F_2G_3-F_3G_2)}{\partial x_1}+\frac{\partial(G_1F_3-G_3F_1)}{\partial x_2}+\frac{\partial(G_2F_1-G_1F_2)}{\partial x_3}$$

applying product rule

$$\nabla\cdot(F\times G)=F_2\frac{\partial G_3}{\partial x_1}+G_3\frac{\partial F_2}{\partial x_1}-F_3\frac{\partial G_2}{\partial x_1}-G_2\frac{\partial F_3}{\partial x_1}+G_1\frac{\partial F_3}{\partial x_2}+F_3\frac{\partial G_1}{\partial x_2}$$
$$-G_3\frac{\partial F_1}{\partial x_2}-F_1\frac{\partial G_3}{\partial x_2}+G_2\frac{\partial F_1}{\partial x_3}+F_1\frac{\partial G_2}{\partial x_3}-G_1\frac{\partial F_2}{\partial x_3}-F_2\frac{\partial G_1}{\partial x_3}$$

$$G\cdot\nabla\times F-F\cdot\nabla\times G=G_1\frac{\partial F_3}{\partial x_2}-G_1\frac{\partial F_2}{\partial x_3}-G_2\frac{\partial F_3}{\partial x_1}+G_2\frac{\partial F_1}{\partial x_3}+G_3\frac{\partial F_2}{\partial x_2}-G_3\frac{\partial F_1}{\partial x_2}$$
$$-F_1\frac{\partial G_3}{\partial x_2}+F_1\frac{\partial G_2}{\partial x_3}+F_2\frac{\partial G_3}{\partial x_1}-F_2\frac{\partial G_1}{\partial x_3}-F_3\frac{\partial G_2}{\partial x_1}+F_3\frac{\partial G_1}{\partial x_2}$$
$$=\nabla\cdot(F\times G)$$

$\boxed{2}$

$$\frac{d}{dt}\int_{V_t} \rho s \, dV \geq -\int_{S_t} \frac{q \cdot n}{T} \, dS$$

Gauss Theorem

$$\frac{d}{dt}\int_{V_t} \rho s \, dV \geq -\int_{V_t} \frac{\nabla q}{T} \, dV$$

Product rule

$$\int_{V_t} \left(\frac{\rho \, ds}{dt} + \frac{s \, d\rho}{dt} + \frac{\nabla q}{T} \right) dV \geq 0$$

$$\frac{\rho \, ds}{dt} + \frac{s \, d\rho}{dt} + \frac{\nabla q}{T} \geq 0$$

$$\rho \dot{s} + \underbrace{\cancel{s \dot{\rho}}}_{0} - \underbrace{\cancel{\frac{K \nabla^2 T}{T}}}_{0} \geq 0$$

mass conservation

$$\frac{K \nabla^2 T}{T} = 0$$

$$\boxed{\rho \dot{s} \geq 0}$$

$\rho \rightarrow$ must be always positive

$s \rightarrow$ entropy always increases, therefore $\dot{s} \geq 0$