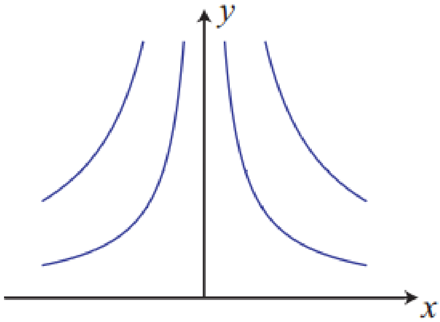
Homework 4.

Task 1.

Consider a fluid stream whose velocity vector coincides with the y axis that impinges on a plane boundary that coincides with the x axis, as shown in the figure below.



(a) Stream function $\Psi(r,\theta) = Ur^2\sin(2\theta) = 2Ur^2\sin(\theta)\cos(\theta)$. As going to Cartesian coordinates $x = r\cos(\theta)\ and\ y = r\sin(\theta)$, the stream function takes the form $\Psi(x,y) = 2Uxy$. Let us compute the velocity field:

$$\begin{cases} u = \dfrac{\partial\Psi(x,y)}{\partial y} = 2Ux \\ v = -\dfrac{\partial\Psi(x,y)}{\partial x} = -2Uy \end{cases}$$

For ideal fluid the point (0, 0) is a stagnation point, so, here the velocity should be 0. Indeed u(0,0)=v(0,0)=0.

From Bernoulli's equation we receive the pressure distribution:

$$\frac{1}{2}\boldsymbol{v}^2 + \frac{p}{\rho} = \frac{p_0}{\rho} \quad\Rightarrow\quad p = p_0 - \frac{\rho}{2}\boldsymbol{v}^2 \quad\Rightarrow\quad p = p_0 - 2\rho U^2(x^2+y^2)$$

where $\boldsymbol{v} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$.

(b) As we consider ideal fluid, the velocity does not depend on time $(\boldsymbol{v}_t = 0)$ and there are no body forces, the Navier-Stokes equations take the view:

$$\begin{cases} \nabla\cdot\boldsymbol{v} = 0 \\ (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \nu\boldsymbol{\nabla}^2\boldsymbol{v} + \dfrac{1}{\rho}\boldsymbol{\nabla}p = 0 \end{cases}$$

Let us check the first equation:

$$\nabla\cdot\boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2U - 2U = 0$$

The second equation has the form:

$$\begin{cases} (\boldsymbol{v}\cdot\boldsymbol{\nabla})u - \nu\boldsymbol{\nabla}^2 u + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ (\boldsymbol{v}\cdot\boldsymbol{\nabla})v - \ \nu\boldsymbol{\nabla}^2 v + \ \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \end{cases}$$

Applying the velocity field to these equations, we receive:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \ \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} = 4U^2 x + 0 - 0 - 4U^2 x = 0;$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \ \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial y} = 4U^2 y + 0 - 0 - 4U^2 y = 0.$$

Which means that the obtained velocity satisfies to Navier-Stokes equations.

For viscous fluid there is a no-slip boundary condition at wall for the velocity which means that $\boldsymbol{v} = 0$ if y=0. This condition cannot be satisfied as velocity component u does not depend on y and $u(x,0) = 2Ux$.

(c) Let us attempt the u velocity component for viscous fluid as $u = 2Uxf'(y)$. As $u = \frac{\partial\Psi(x,y)}{\partial y}$, we can find the stream function: $\Psi(x,y) = \int u dy = \int 2Uxf^{'}(y)dy = 2Uxf(y)$. Now we can obtain the velocity component v: $v = -\frac{\partial\Psi(x,y)}{\partial x} = -2Uf(y)$.

Let us determine boundary conditions for the function $f(y)$. As $v(x,0) = 0, f(0) = 0$. And as $u(x,0) = 0, f^{'}(y) = 0$.

For region sufficiently away from the wall, the viscous effect is negligible and the flow is expected to match with the inviscid flow result. Thus we require:

$$\begin{cases} 2Uxf^{'}(y) \to 2Ux \\ -2Uf(y) \to -2Uy \end{cases} \Rightarrow \begin{cases} f^{'}(y) \to 1 \\ f(y) \to y \end{cases} when\, y \to \infty$$

To sum up, boundary conditions for f:

$$f(0) = f^{'}(y) = 0$$

$$\begin{cases} f^{'}(y) \to 1 \\ f(y) \to y \end{cases} when\ y \to \infty$$

(d) Let us consider the y-momentum equation to get pressure distribution in terms of function f(y):

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \ \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial y} = 4U^2 f(y)f^{'}(y) + \nu\cdot 2Uf^{''}(y) + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial y} = -\rho\left(4U^2 f(y)f^{'}(y) + \nu\cdot 2Uf^{''}(y)\right)$$

Integrating, we receive:

$$p = -\rho(2U^2 f^2 + 2\nu Uf^{'}) + C(x)$$

Recalling that $f(y) \to y$ for large values of y shows that

$$p \to -\rho(2U^2y^2 + 2\nu U) + C(x)$$

which by comparison with the potential-flow pressure, requires:

$$p_0 - 2\rho U^2(x^2 + y^2) = -\rho(2U^2y^2 + 2\nu U) + C(x) \Rightarrow C(x) = p_0 - 2\rho U^2x^2 + 2\rho\nu U.$$

Substituting, we receive the following pressure distribution:

$$p = -2\rho U^2 f^2 - 2\rho\nu U f' + p_0 - 2\rho U^2 x^2 + 2\rho\nu U = p_0 - 2\rho U^2(f^2 + x^2) + 2\rho\nu U(1 - f').$$

(e) Let us consider the x-momentum equation and obtained formula for pressure distribution:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} = 4U^2x(f')^2 - 4U^2xf'' - 2\nu Uxf''' - 4U^2x = 0 \quad \Rightarrow$$

$$\frac{\nu}{2U}f''' - (f')^2 + ff'' + 1 = 0.$$

We have received differential equation for function f. Let us introduce the dimensionless variables:

$$\eta = y\sqrt{\frac{2U}{\nu}} \quad and \quad F(\eta) = \sqrt{\frac{2U}{\nu}}\ f(y).$$

Then the equation takes the view:

$$F''' - (F')^2 + FF'' + 1 = 0$$

where $F' = \frac{\partial F}{\partial \eta}$. Applying received boundary conditions for the function f(y), receive BC for new function F(η):

$$F(0) = F'(0) = 0; \quad F'(\eta) \to 1 \; when \; \eta \to \infty$$

This third-order ODE can be solved numerically.

Task 2.

A quadratic polynomial form for the velocity profile:

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

Boundary conditions:

$$u = 0 \quad if \;\; y = 0$$

$$\begin{cases} u = U \\ \dfrac{\partial u}{\partial y} = 0 \end{cases} \quad if \;\; y = \delta$$

Applying boundary conditions, we receive the system of equations:

$$\begin{cases} \dfrac{u}{U}(y = 0) \; = \; a \; = \; 0; \\ \dfrac{u}{U}(y = \delta) \; = \; a \; + \; b \; + \; c \; = \; 1; \\ \dfrac{\left(\frac{\partial u}{\partial y}\right)}{U} \; (y = \delta) \; = \; b + 2c \; = \; 0. \end{cases}$$

Solving the system, receive unknown parameters: $\begin{cases} a = 0 \\ b = 2 \\ c = -1 \end{cases}$.

These conditions give:

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2.$$

The momentum integral equation reduces to:

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_0}{\rho} \quad \Rightarrow \quad \frac{d}{dx}\int_0^\infty u(U - u)\, dy = \frac{\tau_0}{\rho} \qquad (1)$$

where $U^2\theta = \int_0^\infty u(U - u)\, dy$ - momentum thickness.

Thus, we can compute the momentum thickness:

$$\theta = U^{-2}\int_0^\delta u(U - u)\, dy \; = \; \int_0^\delta \left(\frac{u}{U} - \left(\frac{u}{U}\right)^2\right) dy$$

$$= \int_0^\delta \left(2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 - 4\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^4 + 4\left(\frac{y}{\delta}\right)^3\right) dy = \frac{2}{15}\delta \qquad (2)$$

The shear stress on the surface is given by formula:

$$\tau_0 = \mu\frac{\partial u}{\partial y}|_{y=0} = \mu\frac{\partial u}{\partial y}|_{y=0} = \mu\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)U|_{y=0} = \frac{2U\mu}{\delta}.$$

$$\text{As} \quad \mu = \rho\nu \quad \Rightarrow \quad \frac{\tau_0}{\rho} = \frac{2U\nu}{\delta} \qquad (3).$$

From equations (1)-(3) we receive:

$$\frac{2}{15}U^2\frac{d\delta}{dx} = \frac{2U\nu}{\delta} \quad \Rightarrow \quad \delta d\delta = \frac{15\nu}{U}dx \quad \Rightarrow \quad \delta^2 = \frac{30\nu}{U}x + C.$$

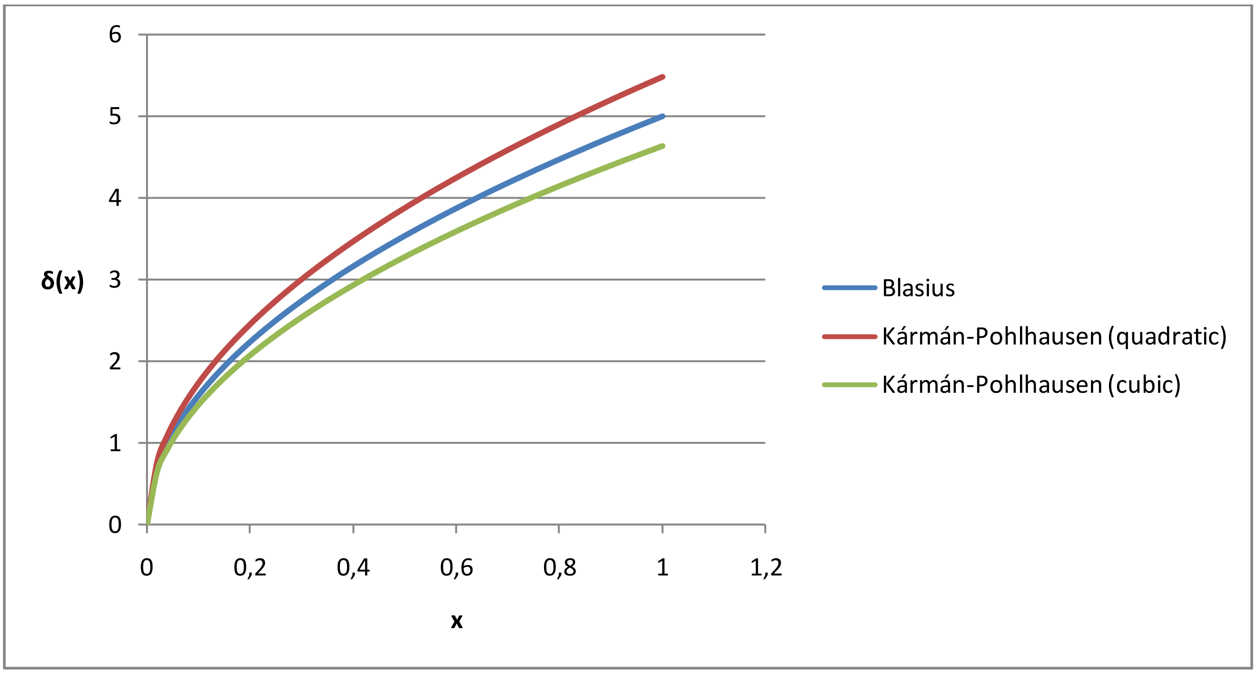
Assuming $\delta(0) = 0$, we receive $\delta = 5.477\sqrt{\frac{\nu x}{U}} = 5.477\frac{x}{\sqrt{Re_x}}$ where $Re_x = \frac{Ux}{\nu}$.

From (2) receive the momentum thickness: $\theta = \frac{2}{15}\delta = 0.7303\frac{x}{\sqrt{Re_x}}$.

Comparing the results with the exact Blasius solution and with the ones obtained assuming a cubic velocity profile:

	$\frac{u}{U}$	δ	θ
Blasius	$f'\left(\frac{y}{\sqrt{\frac{\nu x}{U}}}\right)$	$5\frac{x}{\sqrt{Re_x}}$	$0.664\frac{x}{\sqrt{Re_x}}$
Kármán-Pohlhausen (quadratic)	$2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$	$5.477\frac{x}{\sqrt{Re_x}}$	$0.7303\frac{x}{\sqrt{Re_x}}$
Kármán-Pohlhausen (cubic)	$\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2$	$4.64\frac{x}{\sqrt{Re_x}}$	$0.646\frac{x}{\sqrt{Re_x}}$

The results of the boundary layer thickness δ(x):



The results of the momentum thickness $\theta(x)$: