Theoretical Homework

Coupled Problems

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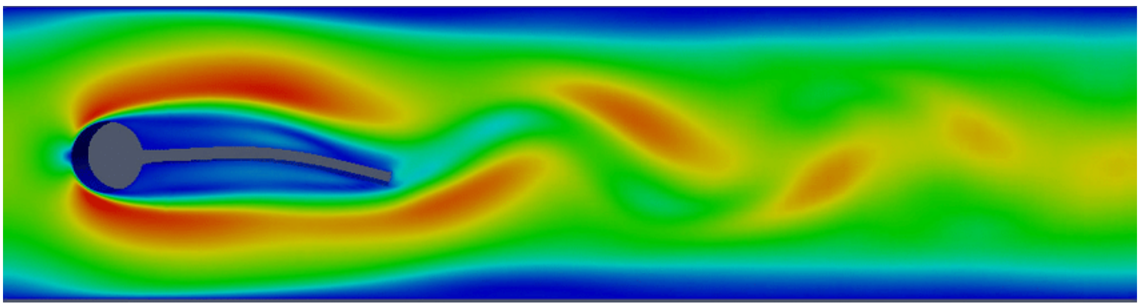

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1 Transmission conditions

1.1 Problem 1

The deflection $v(x)$ of an Euler-Bernoulli beam is governed by the differential equation

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = f$$

where EI is a mechanical property of the beam section and the beam material and f is the distributed load. Assuming for example that the beam is clamped at $x = 0$ and $x = L$, the Principle of Virtual Work (PVW) states that the solution $v(x)$ satisfies

$$EI\int_0^L \frac{\mathrm{d}^2\delta v}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \int_0^L \delta v f$$

for all δv such that $\delta v(0) = \delta v(L) = 0$, $\frac{\mathrm{d}\delta v}{\mathrm{d}x}(0) = \frac{\mathrm{d}\delta v}{\mathrm{d}x}(L) = 0$

- (a) Postulate the space of functions where both v and δv must belong. Justify the answer.
- (b) If $[0, L] = [0, P] \cup (P, L]$, obtain the transmission conditions at P implied by regularity requirements.
- (c) Obtain the transmission conditions at P that follow by imposing in the PVW that the integral is additive.

Solution (a):
In mathematics, a square-integrable function, also called a quadratically integrable function or L^2 function, is a real- or complex-valued measurable function for which the integral of the square of the absolute value is finite. Therefore the right hand side function

$$\int_0^L \delta v f < \infty \longrightarrow \delta v \in L^2$$

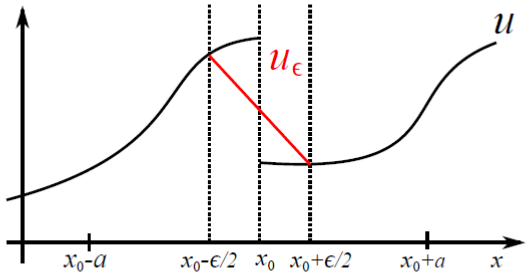
For the left hand side, we can also noticed that the function needs to be bounded, therefore:

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} < \infty \quad \in L^2$$

In dimension $d \in \{2, 3\}$, Embedding of $W^{s,p}(D)$ says that functions in $H^2(D)$ that the functions must not only be bounded, but must also be continuous, therefore:

$$\delta v, v \in H^2$$

Solution (b):
Considering a regularised function v^ϵ for the deflection and dv^ϵ for the first derivative connecting two points separated a distance ϵ across the boundary Γ^P of the partition of Ω.



Consider $x_0 = P$ and $u = v$ for the aim of the problem

$$v = \lim_{\varepsilon \to 0} v^{\varepsilon}$$

$$\left.\frac{dv}{dx}\right|_p = \lim_{\varepsilon \to 0} \left.\frac{dv^{\varepsilon}}{dx}\right|_p$$

Let us assume that

$$\int_{P-a}^{P+a} \frac{dv}{dx} = \lim_{\epsilon \to 0} \int_{P-a}^{P+a} \frac{dv^{\epsilon}}{dx}$$

Because

$$\begin{aligned}\int_{P-a}^{P+a} \frac{dv^{\epsilon}}{dx} &= \int_{P-a}^{P-\epsilon/2} \frac{dv^{\epsilon}}{dx} + \int_{P-\epsilon/2}^{P+\epsilon/2} \frac{dv^{\epsilon}}{dx} + \int_{P+\epsilon/2}^{P+a} \frac{dv^{\epsilon}}{dx} \\ &= \int_{P-a}^{P-\epsilon/2} \frac{dv}{dx} + \epsilon \left[\frac{v(P+\epsilon/2) - v(P-\epsilon/2)}{\epsilon}\right] + \int_{P+\epsilon/2}^{P+a} \frac{dv}{dx} \\ &\underset{\epsilon \to 0}{\longrightarrow} \int_{P-a}^{P} \frac{dv}{dx} + \left[v(P+\epsilon/2) - v(P-\epsilon/2)\right] + \int_{P}^{P+a} \frac{dv}{dx}\end{aligned}$$

the integral of he first derivative of a discontinuous functions makes sense and can be written in terms of

$$[v] = v\left(P^{+}\right) - v\left(P^{-}\right)$$

the jump of v at P However, we have

$$\begin{aligned}\int_{P-a}^{P+a} \left(\frac{dv^{\epsilon}}{dx}\right)^2 = \int_{P-a}^{P-\epsilon/2} \left(\frac{dv}{dx}\right)^2 + \int_{P+\epsilon/2}^{P+a} \left(\frac{dv}{dx}\right)^2 \\ + \epsilon \left[\frac{v(P+\epsilon/2) - v(P-\epsilon/2)}{\epsilon}\right]^2 \underset{\epsilon \to 0}{\longrightarrow} \infty\end{aligned}$$

therefore $v \notin H^1(\Omega)$

Which means, the first transmission condition is:

$$[\![v]\!] = v\left(P^{+}\right) - v\left(P^{-}\right) = 0$$

Now for the first derivative $\frac{dv}{dx}$, lets assume:

$$\frac{du}{dx} = \lim_{\epsilon \to 0} \frac{dv^{\epsilon}}{dx}$$

$$\left.\frac{d^2v}{dx^2}\right|_P = \lim_{\epsilon\to 0}\left.\frac{dv^\epsilon}{dx}\right|_P$$

Repeating the same procedure than before, but substituting v for dv:

$$\int_{P-a}^{P+a}\left(\frac{d^2v^\epsilon}{dx^2}\right) = \int_{P-a}^{P-\epsilon/2}\left(\frac{d^2v}{dx^2}\right) + \int_{P+\epsilon/2}^{P+a}\left(\frac{d^2v}{dx^2}\right) \\ + \epsilon\left[\frac{\frac{dv}{dx}\left(P+\epsilon/2\right) - \frac{dv}{dx}\left(P-\epsilon/2\right)}{\epsilon}\right]$$

The jump of $\frac{dv}{dx}$ around P is defined as:

$$[\![\frac{dv}{dx}]\!] = \frac{dv}{dx}\left(P^+\right) - \frac{dv}{dx}\left(P^-\right)$$

But

$$\int_{P-a}^{P+a}\left(\frac{d^2v^\epsilon}{dx^2}\right)^2 = \int_{P-a}^{P-\epsilon/2}\left(\frac{d^2v}{dx^2}\right)^2 + \int_{P+\epsilon/2}^{P+a}\left(\frac{d^2v}{dx^2}\right)^2 \\ + \epsilon\left[\frac{\frac{dv}{dx}\left(P+\epsilon/2\right) - \frac{dv}{dx}\left(P-\epsilon/2\right)}{\epsilon}\right]^2 \xrightarrow[\epsilon\to 0]{} \infty$$

The function v must be continuous therefore, the second transmission condition yields:

$$[\![\frac{dv}{dx}]\!] = \frac{dv}{dx}\left(P^+\right) - \frac{dv}{dx}\left(P^-\right) = 0$$

Solution (c):

$$\int_\Omega \delta v EI\frac{d^4v}{dx^4} = \int_\Omega \delta v f$$

Integrating by parts yields:

$$-\int_\Omega \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_\Omega \frac{d}{dx}\left(\delta v EI\frac{d^3v}{dx^3}\right) = \int_\Omega \delta v f$$

Applying divergence theorem:

$$-\int_\Omega \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega} \delta v EI\frac{d^3v}{dx^3}n = \int_\Omega \delta v f$$

Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$:
Sub-domain Ω_1:

$$-\int_{\Omega_1} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_1} \delta v EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta v f$$

Splitting the boundary of the sub-domain:

$$-\int_{\Omega_1} \frac{d\delta v}{dx}EI\frac{d^3v}{dx^3} + \int_{\partial\Omega_1\cap\partial\Omega} \delta v EI\frac{d^3v}{dx^3}n_1 + \int_{\partial\Omega_1\cap\Gamma} \delta v EI\frac{d^3v}{dx^3}n_1 = \int_{\Omega_1} \delta v f$$

Sub-domain Ω_2:

$$-\int_{\Omega_2} \frac{d\delta v}{dx} EI \frac{d^3 v}{dx^3} + \int_{\partial\Omega_2} \delta v EI \frac{d^3 v}{dx^3} n_2 = \int_{\Omega_2} \delta v f$$

Splitting the boundary of the sub-domain:

$$-\int_{\Omega_2} \frac{d\delta v}{dx} EI \frac{d^3 v}{dx^3} + \int_{\partial\Omega_2 \cap \partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n_2 + \int_{\partial\Omega_2 \cap \Gamma} \delta v EI \frac{d^3 v}{dx^3} n_2 = \int_{\Omega_2} \delta v f$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$\int_{\partial\Omega_1 \cap \Gamma} \delta v EI \frac{d^3 v}{dx^3} n_1 + \int_{\partial\Omega_2 \cap \Gamma} \delta v EI \frac{d^3 v}{dx^3} n_2 = 0$$

which is simply written as:

$$\int_{\Gamma} \delta v \left((EI)_1 \frac{d^3 v}{dx^3} n_1 + (EI)_2 \frac{d^3 v}{dx^3} n_2 \right) = 0$$

The third transmission condition yields:

$$\left[\!\left[EI \frac{d^3 v}{dx^3} \right]\!\right]_P = (EI)_1 \left(\frac{d^3 v(P-)}{dx^3} \right) - (EI)_2 \left(\frac{d^3 v(P+)}{dx^3} \right) = 0$$

$$\left[\!\left[EI \frac{d^3 v}{dx^3} n \right]\!\right]_\Gamma = 0$$

This represents the equality of shear force on the interface.

Now integrating $-\int_\Omega \frac{d\delta v}{dx} EI \frac{d^3 v}{dx^3} + \int_{\partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n = \int_\Omega \delta v f$ by parts:

$$\int_\Omega \frac{d^2 \delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_\Omega \frac{d}{dx} \left(\frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} \right) + \int_{\partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n = \int_\Omega \delta v f$$

Applying divergence theorem:

$$\int_\Omega \frac{d^2 \delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_{\partial\Omega} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n + \int_{\partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n = \int_\Omega \delta v f$$

Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$:
Sub-domain Ω_1:

$$\int_{\Omega_1} \frac{d^2 \delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_{\partial\Omega_1} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_1 + \int_{\partial\Omega_1} \delta v EI \frac{d^3 v}{dx^3} n_1 = \int_{\Omega_1} \delta v f$$

Splitting the boundary of the sub-domain:

$$\begin{aligned} \int_{\Omega_1} \frac{d^2 \delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_{\partial\Omega_1 \cap \partial\Omega} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_1 - \int_{\partial\Omega_1 \cap \Gamma} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_1 \\ + \int_{\partial\Omega_1 \cap \partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n_1 + \int_{\partial\Omega_1 \cap \Gamma} \delta v EI \frac{d^3 v}{dx^3} n_1 = \int_{\Omega_1} \delta v f \end{aligned}$$

Sub-domain Ω_2:

$$\int_{\Omega_2} \frac{d^2\delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_{\partial\Omega_2} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_2 + \int_{\partial\Omega_2} \delta v EI \frac{d^3 v}{dx^3} n_2 = \int_{\Omega_2} \delta v f$$

Splitting the boundary of the sub-domain:

$$\int_{\Omega_2} \frac{d^2\delta v}{dx^2} EI \frac{d^2 v}{dx^2} - \int_{\partial\Omega_2\cap\partial\Omega} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_2 - \int_{\partial\Omega_2\cap\Gamma} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_2$$
$$+ \int_{\partial\Omega_2\cap\partial\Omega} \delta v EI \frac{d^3 v}{dx^3} n_2 + \int_{\partial\Omega_2\cap\Gamma} \delta v EI \frac{d^3 v}{dx^3} n_2 = \int_{\Omega_2} \delta v f$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$-\int_{o\Omega_1\cap\Gamma} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_1 - \int_{\partial\Omega_2\cap\Gamma} \frac{d\delta v}{dx} EI \frac{d^2 v}{dx^2} n_2 + \int_{\partial\Omega_1\cap\Gamma} \delta v EI \frac{d^3 v}{dx^3} n_1 + \int_{\partial\Omega_2\cap\Gamma} \delta v EI \frac{d^3 v}{dx^3} n_2 = 0$$

which is simply written as:

$$-\int_{\Gamma} \frac{d\delta v}{dx} \left((EI)_1 \frac{d^2 v}{dx^2} n_1 + (EI)_2 \frac{d^2 v}{dx^2} n_2 \right) + \int_{\Gamma} \delta v \left((EI)_1 \frac{d^3 v}{dx^3} n_1 + (EI)_2 \frac{d^3 v}{dx^3} n_2 \right) = 0$$

Notice that the second term is the third transmission condition (which is zero), therefore:

$$\int_{\Gamma} \delta v \left((EI)_1 \frac{d^3 v}{dx^3} n_1 + (EI)_2 \frac{d^3 v}{dx^3} n_2 \right) = 0 \quad \Longrightarrow \quad \left[\!\left[EI \frac{d^3 v}{dx^3} n \right]\!\right]_{\Gamma} = 0$$

The fourth transmission condition yields:

$$\left[\!\left[EI \frac{d^2 v}{dx^2} \right]\!\right]_{P} = (EI)_1 \left(\frac{d^2 v(P-)}{dx^2} \right) - (EI)_2 \left(\frac{d^2 v(P+)}{dx^2} \right) = 0$$

$$\boxed{\left[\!\left[EI \frac{d^2 v}{dx^2} \right]\!\right]_{P} = 0}$$

This represents the equality of bending moments on the interface.

1.2 Problem 2

The Maxwell problem consists in finding a vector field $u : \Omega \longrightarrow \mathbb{R}^3$ such that

$$\begin{aligned} \nu \nabla \times \nabla \times u &= f \text{ in } \Omega \\ \nabla \cdot u &= 0 \text{ in } \Omega \\ n \times u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $\nu > 0$, f is a divergence free force field and n the unit external normal. Equation $\nabla \cdot u = 0$ is in fact redundant.

- (a) Write a variational statement of the problem. Postulate the space of functions where u must belong. Justify the answer.
- (b) If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ implied by regularity requirements.
- (c) Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

Solution (a):
Multiplying the Maxwell problem and integrating over the domain:

$$\int_\Omega \delta u \cdot (\nu \nabla \times \nabla \times u) = \int_\Omega \delta u \cdot f$$

Using the following identities:

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B) \quad \rightarrow \quad A \cdot (\nabla \times B) = (\nabla \times A) \cdot B - \nabla \cdot (A \times B)$$

Where:

$$A = \delta u \quad \Longleftrightarrow \quad B = \nabla \times u$$

Therefore:

$$\int_\Omega \nu A \cdot (\nabla \times B) = \int_\Omega \nu (\nabla \times A) \cdot B - \int_\Omega \nu \nabla \cdot (A \times B)$$

Applying the divergence theorem:

$$\int_\Omega \nu A \cdot (\nabla \times B) = \int_\Omega \nu (\nabla \times A) \cdot B - \int_\Gamma \nu n \cdot (A \times B)$$

Applying the following identity:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

We get the following:

$$\int_\Omega \nu A \cdot (\nabla \times B) = \int_\Omega \nu (\nabla \times A) \cdot B - \int_\Gamma \nu B \cdot (n \times A)$$

Now substituting:

$$A = \delta u \quad \Longleftrightarrow \quad B = \nabla \times u$$

Yields:

$$\int_\Omega \delta u \cdot (\nu \nabla \times \nabla \times u) = \int_\Omega (\nabla \times u) \cdot (\nabla \times \delta u) - \int_\Gamma \nu (\nabla \times u) \cdot (n \times \delta u)$$

Where $n \times \delta u = 0$ on $\partial\Omega$, therefore the weak form is:

$$\int_\Omega (\nabla \times u) \cdot (\nu \nabla \times \delta u) = \int_\Omega \delta u \cdot f$$

Where the space of functions for u and δu are defined by $\mathcal{H}^{\text{curl}}(\Omega)$ which is the space of vector functions of dimension (d) that are defined in Ω such that a function and curl are square integrable (L^2), therefore:

$$u \in \mathcal{H}^{\text{curl}}(\Omega) \quad such \;\; that \; (n \times u)|_{\partial\Omega} = 0$$
$$\delta u \in \mathcal{H}^{\text{curl}}(\Omega) \quad such \;\; that \; (n \times \delta u)|_{\partial\Omega} = 0$$
$$\mathcal{H}^{\text{curl}}(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d | \boldsymbol{u} \in [\mathcal{L}_2(\Omega)]^d, \boldsymbol{\nabla} \times \boldsymbol{u} \in [\mathcal{L}_2(\Omega)]^d \right\}$$

Solution (b):
The following condition must be fulfilled:

$$\int_\Omega |\nabla \times u|^2 < \infty$$

Considering an split of the definition of the problem to overcome the discontinuities $n \times u$ may present:

- $x \in \Omega_1$:
$$\nabla \times u$$
- $x \in x_0 - \epsilon \leq x \leq x_0 + \epsilon$:
$$\frac{1}{2\varepsilon}\left[\boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o + \varepsilon\right) - \boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o - \varepsilon\right)\right]$$
- $x \in \Omega_2$:
$$\nabla \times u$$

Therefore the square integral is calculated as:

$$\begin{aligned}\int_\Omega (\boldsymbol{\nabla} \times \boldsymbol{u})^2 &= \int_{\Omega_1} (\boldsymbol{\nabla} \times \boldsymbol{u})^2 + \int_{\boldsymbol{x}_o+\varepsilon}^{\boldsymbol{x}_o-\varepsilon} \left(\frac{1}{2\varepsilon}\left[\boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o + \varepsilon\right) - \boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o - \varepsilon\right)\right]\right)^2 + \int_{\Omega_2} (\boldsymbol{\nabla} \times \boldsymbol{u})^2 \\ &= \int_{\Omega_1} (\boldsymbol{\nabla} \times \boldsymbol{u})^2 + \frac{1}{2\varepsilon}\left[\boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o + \varepsilon\right) - \boldsymbol{n} \times \boldsymbol{u}\left(\boldsymbol{x}_o - \varepsilon\right)\right]^2 + \int_{\Omega_2} (\boldsymbol{\nabla} \times \boldsymbol{u})^2\end{aligned}$$

Noticed that as $\varepsilon \to 0 = \infty$, therefore if $n \times u$ is discontinuous, $\nabla \times u$ is not square integrable. This means, $u \notin \mathcal{H}^{\text{curl}}$.

Therefore, the first transmission condition yields:

$$[\![n \times u]\!]_\Gamma = 0$$

Solution (c):
Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an

interface $\Gamma = \Omega_1 \cap \Omega_2$:
Sub-domain Ω_1:

$$\int_{\Omega_1} (\boldsymbol{\nabla} \times \delta \boldsymbol{u}) \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_1 \cap \partial\Omega} \nu(\nabla \times u) \cdot (n_1 \times \delta u) - \int_{\partial\Omega_1 \cap \Gamma} \nu(\nabla \times u) \cdot (n_1 \times \delta u) = \int_{\Omega_1} \delta \boldsymbol{u} \cdot \boldsymbol{f}$$

Sub-domain Ω_2:

$$\int_{\Omega_2} (\boldsymbol{\nabla} \times \delta \boldsymbol{u}) \cdot (\nu \boldsymbol{\nabla} \times \boldsymbol{u}) - \int_{\partial\Omega_2 \cap \partial\Omega} \nu(\nabla \times u) \cdot (n_2 \times \delta u) - \int_{\partial\Omega_2 \cap \Gamma} \nu(\nabla \times u) \cdot (n_2 \times \delta u) = \int_{\Omega_2} \delta \boldsymbol{u} \cdot \boldsymbol{f}$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$\int_{\partial\Omega_1 \cap \Gamma} \nu(\nabla \times u) \cdot (n_1 \times \delta u) + \int_{\partial\Omega_2 \cap \Gamma} \nu(\nabla \times u) \cdot (n_2 \times \delta u) = 0$$

Leading to have the second transmission condition defined as:

$$[\![\nu(\nabla \times u \times n)]\!]_\Gamma = 0$$

1.3 Problem 3

The Navier equations for an elastic material can be written in three different ways:

$$-2\mu\nabla \cdot (\varepsilon(\boldsymbol{u})) - \lambda\nabla(\nabla \cdot \boldsymbol{u}) = \rho\boldsymbol{b}$$
$$-\mu\Delta\boldsymbol{u} - (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = \rho\boldsymbol{b}$$
$$\mu\nabla \times (\nabla \times \boldsymbol{u}) - (\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) = \rho\boldsymbol{b}$$

where u is the displacement field, $\varepsilon(u)$ the symmetric part of $\nabla u, \lambda$ and μ the Lamé coefficients, ρ the density of the material and b the body forces. Let us assume that $u = 0$ on $\partial\Omega$

- (a) Write down the variational form of the previous equations in the appropriate functional spaces.
- (b) If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

Solution (a):

- $-2\mu\nabla \cdot (\varepsilon(\boldsymbol{u})) - \lambda\nabla(\nabla \cdot \boldsymbol{u}) = \rho\boldsymbol{b}$

To obtain the weak form, we first have to multiply by a test function δu and integrate over the domain:

$$\int_\Omega \delta\boldsymbol{u}\cdot(-2\mu\boldsymbol{\nabla}\cdot\varepsilon(\boldsymbol{u})) - \int_\Omega \delta\boldsymbol{u}\cdot\lambda\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = \int_\Omega \delta\boldsymbol{u}\cdot\rho\boldsymbol{b}$$

Integrating by parts and applying the divergence theorem:
First term:

$$\int_\Omega \delta\boldsymbol{u}\cdot(-2\mu\boldsymbol{\nabla}\cdot\varepsilon(\boldsymbol{u})) = 2\mu\int_\Omega \nabla\delta u : \varepsilon - 2\mu\int_{\partial\Omega}(\varepsilon\delta u)\cdot n$$

Second term:

$$-\int_\Omega \delta\boldsymbol{u}\cdot\lambda\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = \lambda\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot\delta u) - \lambda\int_{\partial\Omega}\delta u(\nabla\cdot u)\cdot n$$

Therefore the weak form yields:

$$2\mu\int_\Omega \nabla\delta u : \varepsilon - 2\mu\int_{\partial\Omega}(\varepsilon\delta u)\cdot n + \lambda\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot\delta u) - \lambda\int_{\partial\Omega}\delta u(\nabla\cdot u)\cdot n = \int_\Omega \delta u\cdot\rho b$$

Since $u = 0$ on $\partial\Omega(\delta u = 0)$, the weak form can be simplified to:

$$2\mu\int_\Omega \nabla\delta u : \varepsilon + \lambda\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot\delta u) = \int_\Omega \delta u\cdot\rho b$$

- $-\mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \rho\boldsymbol{b}$

To obtain the weak form, we first have to multiply by a test function δu and integrate over the domain:

$$\int_\Omega -\mu\delta u\cdot\nabla\cdot\nabla\boldsymbol{u} - \int_\Omega(\lambda+\mu)\delta u\cdot\nabla(\nabla\cdot\boldsymbol{u}) = \int_\Omega \delta u\cdot\rho\boldsymbol{b}$$

Integrating by parts and applying the divergence theorem:
First term:

$$\int_\Omega -\mu\delta u\cdot\nabla\cdot\nabla\boldsymbol{u} = \mu\int_\Omega \nabla\delta u : \nabla u - \mu\int_{\partial\Omega}(\nabla u\delta u)\cdot n$$

Second term:

$$-\int_\Omega(\lambda+\mu)\delta u\cdot\nabla(\nabla\cdot\boldsymbol{u}) = (\lambda+\mu)\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot u) - (\lambda+\mu)\int_{\partial\Omega}\delta u(\nabla\cdot u)\cdot n$$

Therefore the weak form yields:

$$\mu\int_\Omega \nabla\delta u : \nabla u - \mu\int_{\partial\Omega}(\nabla u\delta u)\cdot n + (\lambda+\mu)\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot u) - (\lambda+\mu)\int_{\partial\Omega}\delta u(\nabla\cdot u)\cdot n$$

Since $u = 0$ on $\partial\Omega(\delta u = 0)$, the weak form can be simplified to:

$$\mu\int_\Omega \nabla\delta u : \nabla u - (\lambda+\mu)\int_\Omega(\nabla\cdot\delta u)(\nabla\cdot u) = \int_\Omega \delta u\cdot\rho b$$

- $\mu\nabla \times (\nabla \times \boldsymbol{u}) - (\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) = \rho\boldsymbol{b}$

Vector calculus identity:

$$\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \nabla^2 u$$

Substituting we have:

$$\mu\nabla(\nabla \cdot u) - \mu\nabla^2 u - (\lambda + 2\mu)\nabla(\nabla \cdot u) = \rho b$$

This proves that it can be used the same expression as before.

Now without considering this identity we have that:

$$\int_\Omega \delta u \cdot (\mu\nabla \times \nabla \times u) - \int_\Omega \delta u \cdot (\lambda + 2\mu)\nabla(\nabla \cdot u) = \int_\Omega \delta u \cdot \rho b$$

For the integration by parts:

$$\delta u \cdot (\nabla \times \nabla \times u) = (\nabla \times \delta u) \cdot (\nabla \times u) - \nabla \cdot (\delta u \times \nabla \times u)$$

$$\nabla \cdot (\delta u(\nabla \cdot u)) = \delta u \cdot \nabla(\nabla \cdot u) + (\nabla \cdot \delta u)(\nabla \cdot u)$$

Hence, the weak form of the problem yields:

$$\int_\Omega (\nabla \times \delta u) \cdot (\mu\nabla \times u) - \int_\Omega \nabla \cdot (\mu\delta u \times \nabla \times u) + \int_\Omega (\lambda + 2\mu)(\nabla \cdot \delta u)(\nabla \cdot u) - \int_\Omega (\lambda + 2\mu)\nabla \cdot (\delta u(\nabla \cdot u)) = \int_\Omega \delta u \cdot \rho b$$

Applying divergence theorem to the second and fourth terms on the LHS yields:

$$\int_\Omega (\nabla\times\delta u)\cdot(\mu\nabla\times u) - \int_{\partial\Omega} (\mu\nabla\times u)\cdot(n\times\delta u) + \int_\Omega (\lambda+2\mu)(\nabla\cdot\delta u)(\nabla\cdot u) - \int_{\partial\Omega} (\lambda+2\mu)(\delta u(\nabla\cdot u))\cdot n = \int_\Omega \delta u\cdot\rho b$$

Since $u = 0$ on $\partial\Omega(\delta u = 0)$, the weak form can be simplified to:

$$\int_\Omega (\nabla \times \delta u) \cdot (\mu\nabla \times u) + \int_\Omega (\lambda + 2\mu)(\nabla \cdot \delta u)(\nabla \cdot u) = \int_\Omega \delta u \cdot \rho b$$

Solution(b):

- First equation

Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$:

Sub-domain Ω_1:

$$\int_{\Omega_1} 2\mu\nabla\delta\boldsymbol{u} : \varepsilon - \int_{\partial\Omega_1\cap\partial\Omega} 2\mu(\varepsilon\delta\boldsymbol{u}) \cdot \boldsymbol{n}_1 - \int_{\boldsymbol{\partial}\Omega_1\cap\Gamma} 2\mu(\varepsilon\delta\boldsymbol{u}) \cdot \boldsymbol{n}_1 + \int_{\Omega_1} \lambda(\nabla \cdot \delta\boldsymbol{u})(\nabla \cdot \boldsymbol{u}) - \int_{\partial\Omega_1\cap\partial\Omega} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) \cdot \boldsymbol{n}_1 - \int_{\partial\Omega_1\cap\Gamma} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{u})) \cdot \boldsymbol{n}_1 = \int_{\Omega_1} \delta\boldsymbol{u} \cdot \rho\boldsymbol{b}$$

Sub-domain Ω_2:

$$\int_{\Omega_2} 2\mu\nabla\delta\boldsymbol{u} : \varepsilon - \int_{\partial\Omega_2\cap\partial\Omega} 2\mu(\varepsilon\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 - \int_{\boldsymbol{\partial}\Omega_2\cap\Gamma} 2\mu(\varepsilon\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 + \int_{\Omega_2} \lambda(\nabla\cdot\delta\boldsymbol{u})(\nabla\cdot\boldsymbol{u})$$
$$- \int_{\partial\Omega_2\cap\partial\Omega} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 - \int_{\partial\Omega_2\cap\Gamma} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = \int_{\Omega_2} \delta\boldsymbol{u}\cdot\rho\boldsymbol{b}$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$\int_{\boldsymbol{\partial}\Omega_1\cap\Gamma} 2\mu(\varepsilon\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 + \int_{\boldsymbol{\partial}\Omega_2\cap\Gamma} 2\mu(\varepsilon\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 = 0$$
$$\int_{\partial\Omega_1\cap\Gamma} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 + \int_{\partial\Omega_2\cap\Gamma} \lambda(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = 0$$

Therefore, the transmission conditions yields:

$$[\![\mu\varepsilon\cdot n]\!]_\Gamma = 0$$
$$[\![\lambda(\nabla\cdot u)n]\!]_\Gamma = 0$$

- Second equation

Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$:

Sub-domain Ω_1:

$$\int_{\Omega_1} \mu\nabla\delta\boldsymbol{u} : \boldsymbol{\nabla u} - \int_{\partial\Omega_1\cap\partial\Omega} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 - \int_{\partial\Omega_1\cap\Gamma} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 + \int_{\Omega_1} (\lambda+\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$
$$- \int_{\partial\Omega_1\cap\partial\Omega} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 - \int_{\partial\Omega_1\cap\Gamma} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 = \int_{\Omega_1} \delta\boldsymbol{u}\cdot\rho\boldsymbol{b}$$

Sub-domain Ω_2:

$$\int_{\Omega_2} \mu\nabla\delta\boldsymbol{u} : \boldsymbol{\nabla u} - \int_{\partial\Omega_2\cap\partial\Omega} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 - \int_{\partial\Omega_2\cap\Gamma} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 + \int_{\Omega_2} (\lambda+\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$
$$- \int_{\partial\Omega_2\cap\partial\Omega} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 - \int_{\partial\Omega_2\cap\Gamma} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = \int_{\Omega_2} \delta\boldsymbol{u}\cdot\rho\boldsymbol{b}$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$\int_{\partial\Omega_1\cap\Gamma} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_1 + \int_{\partial\Omega_2\cap\Gamma} \mu(\boldsymbol{\nabla u}\delta\boldsymbol{u})\cdot\boldsymbol{n}_2 = 0$$
$$\int_{\partial\Omega_1\cap\Gamma} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 + \int_{\partial\Omega_2\cap\Gamma} (\lambda+\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = 0$$

Therefore, the transmission conditions yields:

$$[\![\mu(\nabla u)\cdot n]\!]_\Gamma = 0$$
$$[\![(\lambda+\mu)(\nabla\cdot u)n]\!]_\Gamma = 0$$

- Third equation

Considering a domain which is composed of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$ with an interface $\Gamma = \Omega_1 \cap \Omega_2$:

Sub-domain Ω_1:

$$\begin{aligned}\int_{\Omega_1} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\mu\boldsymbol{\nabla}\times\boldsymbol{u}) - \int_{\partial\Omega_1\cap\partial\Omega}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_1\times\delta\boldsymbol{u}) - \int_{\partial\Omega_1\cap\Gamma}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_1\times\delta\boldsymbol{u})\\ + \int_{\Omega_1}(\lambda+2\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \int_{\partial\Omega_1\cap\partial\Omega}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1\\ - \int_{\partial\Omega_1\cap\Gamma}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 = \int_{\Omega_1}\delta\boldsymbol{u}\cdot\rho\boldsymbol{b}\end{aligned}$$

Sub-domain Ω_2:

$$\begin{aligned}\int_{\Omega_2} (\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\mu\boldsymbol{\nabla}\times\boldsymbol{u}) - \int_{\partial\Omega_2\cap\partial\Omega}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_2\times\delta\boldsymbol{u}) - \int_{\partial\Omega_2\cap\Gamma}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_2\times\delta\boldsymbol{u})\\ + \int_{\Omega_2}(\lambda+2\mu)(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u})(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \int_{\partial\Omega_2\cap\partial\Omega}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2\\ - \int_{\partial\Omega_2\cap\Gamma}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = \int_{\Omega_2}\delta\boldsymbol{u}\cdot\rho\boldsymbol{b}\end{aligned}$$

Considering the additive splitting, summing the resultant equations from both sub-domains must lead to the same equation before we split the domain into sub-domains. This means the extra terms that arise from the interface boundary must be equal to zero:

$$\int_{\partial\Omega_1\cap\Gamma}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_1\times\delta\boldsymbol{u}) + \int_{\partial\Omega_2\cap\Gamma}(\mu\boldsymbol{\nabla}\times\boldsymbol{u})\cdot(\boldsymbol{n}_2\times\delta\boldsymbol{u}) = 0$$

Which yields as:

$$\int_\Gamma \delta\boldsymbol{u}\cdot(\mu\nabla\times\boldsymbol{u}\times\boldsymbol{n}_1 + \mu\boldsymbol{\nabla}\times\boldsymbol{u}\times\boldsymbol{n}_2) = 0$$

And:

$$\int_{\partial\Omega_1\cap\Gamma}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_1 + \int_{\partial\Omega_2\cap\Gamma}(\lambda+2\mu)(\delta\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u}))\cdot\boldsymbol{n}_2 = 0$$

Therefore, the transmission conditions yields:

$$[\![\mu\nabla\times u\times n]\!]_\Gamma = 0$$
$$[\![(\lambda+2\mu)(\nabla\cdot u)n]\!]_\Gamma = 0$$

2 Domain decomposition methods

2.1 Problem 1

Consider Problem 1 of Section 1. Let $[0, L] = [0, L_1] \cup [L_2, L]$, with $L_2 < L_1$

- (a) Write down an iteration-by-subdomain scheme based on a Schwarz additive domain decomposition method.
- (b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

Solution (a):
Considering the Euler-Bernoulli beam, the Schwarz additive domain decomposition method with a Jacobi scheme yields as follows:

Sub-domain Ω_1 :

$$\begin{aligned} EI\frac{d^4 v_1^{(k)}}{dx^4} &= f \longrightarrow in \quad \Omega_1 \\ v_1^{(k)} &= 0 \longrightarrow on \quad \Gamma_1 \\ \frac{dv_1^{(k)}}{dx} &= 0 \longrightarrow on \quad \Gamma_1 \\ v_1^{(k)} = v_2^{(k-1)} &\longrightarrow on \quad \Gamma_{12} \\ \frac{dv_1^{(k)}}{dx} = \frac{dv_2^{(k-1)}}{dx} &\longrightarrow on \quad \Gamma_{12} \end{aligned}$$

Sub-domain Ω_2 :

$$\begin{aligned} EI\frac{d^4 v_2^{(k)}}{dx^4} &= f \longrightarrow in \quad \Omega_2 \\ v_2^{(k)} &= 0 \longrightarrow on \quad \Gamma_2 \\ \frac{dv_2^{(k)}}{dx} &= 0 \longrightarrow on \quad \Gamma_2 \\ v_2^{(k)} = v_1^{(k-1)} &\longrightarrow on \quad \Gamma_{21} \\ \frac{dv_2^{(k)}}{dx} = \frac{dv_1^{(k-1)}}{dx} &\longrightarrow on \quad \Gamma_{21} \end{aligned}$$

Solution (b):
The system of equations yields:

$$Au = b$$

With a Galerkin formulation:

$$A = EI\int_0^L \frac{d^2N}{dx^2}\frac{d^2N}{dx^2}$$

$$b = \int_0^L Nf$$

Therefore the matrix version of the Schwarz method for this problem yields:

$$\begin{bmatrix} A_{11} & A_{1\Gamma_{12}} \\ A_{\Gamma_{12}1} & A_{\Gamma_{12}\Gamma_{12}} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_{1\Gamma_{12}} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_{\Gamma_{12}} \end{bmatrix} \longrightarrow Sub-domain \quad \Omega_1$$

$$\begin{bmatrix} A_{22} & A_{2\Gamma_{21}} \\ A_{\Gamma_{21}2} & A_{\Gamma_{21}\Gamma_{21}} \end{bmatrix} \begin{bmatrix} u_2^{(k)} \\ u_{2\Gamma_{21}} \end{bmatrix} = \begin{bmatrix} b_2 \\ b_{\Gamma_{21}} \end{bmatrix} \longrightarrow Sub-domain \quad \Omega_2$$

Where:

$$u_{1\Gamma_{12}} = u_2^{(k-1)} \quad on \quad L_1$$

$$u_{2\Gamma_{21}} = u_1^{(k-1)} \quad on \quad L_2$$

2.2 Problem 2

Consider Problem 2 of Section 1. Let Γ be a surface that intersects Ω

- (a) Write down an iteration-by-subdomain scheme based on the Dirichlet-Neumann coupling.
- (b) Obtain the expression of the Steklov-Poincaré operator of the problem.
- (c) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

Solution (a):

Sub-domain Ω_1 :

$$\begin{aligned} \nu\nabla\times\nabla\times u_1^{(k)} &= f_1 \longrightarrow in \;\; \Omega_1 \\ \nabla\cdot u_1^{(k)} &= 0 \longrightarrow in \;\; \Omega_1 \\ n_1\times u_1^{(k)} &= 0 \longrightarrow on \;\; \Gamma_1 \\ n_1\times(\nabla\times u_1^{(k)}) &= n_1\times(\nabla\times u_2^{(k-1)}) \longrightarrow on \;\; \Gamma_{12} \end{aligned}$$

Sub-domain Ω_2 :

$$\begin{aligned} \nu\nabla\times\nabla\times u_2^{(k)} &= f_2 \longrightarrow in \;\; \Omega_2 \\ \nabla\cdot u_2^{(k)} &= 0 \longrightarrow in \;\; \Omega_2 \\ n_2\times u_2^{(k)} &= 0 \longrightarrow on \;\; \Gamma_2 \\ n_2\times u_2^{(k)} &= n_2\times u_1^{(k-1)} \longrightarrow on \;\; \Gamma_{21} \end{aligned}$$

Solution(b):

Let $u_i = u_i^0 + \tilde{u}_i$ for $i = 1, 2$ with:

Sub-domain Ω_i :

$$\begin{aligned} \nu\nabla\times\nabla\times u_i^0 &= f_i \longrightarrow in \;\; \Omega_i \\ \nabla\cdot u_i^0 &= 0 \longrightarrow in \;\; \Omega_i \\ n_i\times u_i^0 &= 0 \longrightarrow on \;\; \Gamma_i \\ n_i\times u_i^0 &= 0 \longrightarrow on \;\; \Gamma_{12} \end{aligned}$$

$$\begin{aligned} \nu\nabla\times\nabla\times u_i^0 &= f_i \longrightarrow in \;\; \Omega_i \\ \nabla\cdot \tilde{u}_i &= 0 \longrightarrow in \;\; \Omega_i \\ n_i\times \tilde{u}_i &= 0 \longrightarrow on \;\; \Gamma_i \\ n_i\times \tilde{u}_i &= \varphi \longrightarrow on \;\; \Gamma_{12} \end{aligned}$$

Where the unknown φ, must satisfy the second transmission condition:

$$n\times(\nabla\times u_1) = n\times(\nabla\times u_2) \longrightarrow n\times(\nabla\times(u_1^0+\tilde{u}_1)) = n\times(\nabla\times(u_2^0+\tilde{u}_2))$$

Rearranging terms we obtain:

$$\underbrace{n\times(\nabla\times\tilde{u}_1) - n\times(\nabla\times\tilde{u}_2)}_{\mathcal{S}} = \underbrace{n\times(\nabla\times u_2^0) - n\times(\nabla\times u_1^0)}_{\mathcal{G}}$$

$$\mathcal{S}\varphi = \mathcal{G}$$

Solution (c):
The matrix version yields:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} & 0 \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & A_{\Gamma I}^{(2)} & A_{II}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ u_{\Gamma_{12}} \\ u_I^{(2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{\Gamma_{12}} \\ f_2 \end{bmatrix}$$

Neumann problem for the sub-domain Ω_1, can be written as follows:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{(1)\ [k]} \\ u_{\Gamma_{12}}^{[k]} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{\Gamma_{12}} - A_{I\Gamma}^{(2)} u_I^{(2)\ [k-1]} - A_{\Gamma\Gamma}^{(2)} u_{\Gamma_{12}}^{[k-1]} \end{bmatrix}$$

The Dirichlet problem for the subdomain Ω_2, can be written as follows:

$$A_{II}^{(2)} u_I^{(2)\ [k]} = f_2 - A_{I\Gamma}^{(2)} u_\Gamma^{[k-1]}$$

2.3 Problem 3

Consider the problem of finding $u : \Omega \longrightarrow \mathbb{R}$ such that

$$-k\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

where $k > 0$. Let Γ be a surface crossing Ω

- (a) Write down an iteration-by-subdomain scheme based on the Dirichlet-Robin coupling.
- (b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.
- (c) Obtain the Schur complement as discrete version of the Steklov-Poincaré operator.
- (d) Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section (a).

Solution (a):

Sub-domain Ω_1 : (Dirichlet)

$$-k\nabla^2 u_1^{[k]} = f_1 \longrightarrow in \quad \Omega_1$$
$$u_1 = 0 \longrightarrow on \quad \Gamma_1$$
$$u_1^{[k]} = u_2^{[k-1]} \longrightarrow on \quad \Gamma_{12}$$

Sub-domain Ω_2 : (Robin)

$$-k\nabla^2 u_2^{[k]} = f_2 \longrightarrow in \;\; \Omega_2$$
$$u_2 = 0 \longrightarrow in \;\; \Omega_2$$
$$k\frac{\partial u_2^{[k]}}{\partial n} + \gamma_2 u_2^{[k]} = k\frac{\partial u_1^{[k-1]}}{\partial n} + \gamma_1 u_1^{[k-1]} \longrightarrow on \;\; \Gamma_{12}$$

Solution (b):

Sub-domain Ω_1: (Dirichlet)

$$\int_{\Omega_1} k\nabla\delta u_1 \nabla u_1 = \int_{\Omega_1} \delta u_1 f_1 \longrightarrow in \;\; \Omega_1$$

Sub-domain Ω_2: (Robin)

$$\int_{\Omega_2} k\nabla\delta u_2 \nabla u_2 - \int_{\Gamma_{12}} k\delta u_2 \nabla u_2 \cdot n = \int_{\Omega_1} \delta u_2 f_2$$

$$\int_{\Omega_2} k\nabla\delta u_2 \nabla u_2 - \int_{\Gamma_{12}} k\delta u_2 \frac{\partial u_2}{\partial n} = \int_{\Omega_1} \delta u_2 f_2$$

$$\int_{\Omega_2} k\nabla\delta u_2 \nabla u_2 + \int_{\Gamma_{12}} k\gamma_2 \delta u_2 u_2 = \int_{\Omega_1} \delta u_2 f_2 + \int_{\Gamma_{12}} k\delta u_1 \frac{\partial u_1}{\partial n} + \int_{\Gamma_{12}} k\gamma_1 \delta u_1 u_1 \longrightarrow in \;\; \Omega_2$$

The matrix version yields:

$$A_{II}^{(1)} u_I^{(1)\;[k]} = F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{[k-1]}$$

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} + \gamma_2 M_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(2)\;[k]} \\ u_{\Gamma_{12}}^{[k]} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_{\Gamma_{12}} - (A_{\Gamma I}^{(1)} - \gamma_1 M_{\Gamma I}^{(1)}) u_I^{(1)\;[k-1]} - (A_{\Gamma\Gamma}^{(1)} - \gamma_1 M_{\Gamma\Gamma}^{(1)}) u_{\Gamma_{12}}^{[k-1]} \end{bmatrix}$$

Where M is the mass matrix.

Solution (c):

$$-k\Delta u_i^0 \longrightarrow in \;\; \Omega_i$$
$$u_i^0 = 0 \longrightarrow on \;\; \partial\Omega_i$$
$$u_i^0 = 0 \longrightarrow on \;\; \partial\Gamma$$

$$-k\Delta \tilde{u}_i \longrightarrow in \;\; \Omega_i$$
$$\tilde{u}_i = 0 \longrightarrow on \;\; \partial\Omega_i$$
$$\tilde{u}_i = \phi \longrightarrow on \;\; \partial\Gamma$$

Where:

$$u_i = u_i^0 + \tilde{u}_i$$

Therefore:

$$k_1 \frac{\partial \tilde{u}_i}{\partial n} - k_2 \frac{\partial \tilde{u}_i}{\partial n} = \underbrace{-k_1 \frac{\partial u_i^0}{\partial n} + k_2 \frac{\partial u_i^0}{\partial n}}_{G}$$

And:

$$u_1^0 = A_{11}^{-1} F_1 \qquad \tilde{u}_1 = -A_{11}^{-1}(A_{1\Gamma} U_\Gamma)$$
$$u_2^0 = A_{22}^{-1} F_2 \qquad \tilde{u}_1 = -A_{22}^{-1}(A_{2\Gamma} U_\Gamma)$$

Therefore recalling that $u_i = u_i^0 + \tilde{u}_i$: Sub-domain Ω_1:

$$u_I^{(1)\ [k]} = A_{II}^{(1)-1}(F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{[l]})$$

Sub-domain Ω_2:

$$u_I^{(2)\ [k]} = A_{II}^{(2)-1}(F_2 - A_{I\Gamma}^{(2)} u_\Gamma^{[l]})$$

Performing a matrix-vector multiplication of the second row, and using the second sub-domain Ω_2, we obtain:

$$(-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)}) U_\Gamma^{[k]} = F_\Gamma - A_{\Gamma I}^{(1)} u_I^{(1)\ [k-1]} - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{[k-1]} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$$

Substituting the first sub-domain Ω_1, we obtain:

$$(-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)}) U_\Gamma^{[k]} = F_\Gamma - A_{\Gamma I}^{(1)} A_{II}^{(1)} (F_1 - A_{I\Gamma} u_\Gamma^{[l]}) - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{[k-1]} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$$

Simplifying yields:

$$\underbrace{(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)})}_{S} \underbrace{u_\Gamma^{[k]}}_{U_\Gamma} = \underbrace{F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1}_{G}$$

$$S U_\Gamma = G$$

Where S is the Schur complement:

$$(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)})$$

Solution (d):

To find the preconditioner for the Schur complement we define the following:

$$S = S_1 + S_2$$

Where:

$$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)}$$
$$S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)}$$

And we define G as:

$$G = F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1$$

Now we have that:

$$(S_1 + S_2) u_\Gamma = G$$

Therefore:

$$S_2 u_\Gamma = G - S_1 u_\Gamma \longrightarrow S_2 u_\Gamma^{[k]} = G - (S - S_2) u_\Gamma^{[k-1]}$$
$$S_2 u_\Gamma^{[k]} = G - S u_\Gamma^{[k-1]} + S_2 u_\Gamma^{[k-1]} \longrightarrow u_\Gamma^{[k]} = S_2^{-1} G - S_2^{-1} S u_\Gamma^{[k-1]} + \underbrace{S_2^{-1} S_2}_{I} u_\Gamma^{[k-1]}$$

$$u_\Gamma^{[k]} = u_\Gamma^{[k-1]} + \underbrace{S_2^{-1}}_{\text{Preconditioner}} (G - S u_\Gamma^{[k-1]})$$

3 Coupling of heterogeneous problems

3.1 Problem 1

Consider the beam described in Problem 1 of Section 1 . Apart from being clamped at $x = 0$ and $x = L$, the beam is supported on an elastic wall that occupies the square $[0, L] \times [-L, 0]$, where $y = 0$ corresponds to the beam axis. The wall is clamped everywhere except on the upper wall, where the beam is. The wall displacements in the x - and y -directions are u and v, respectively, and the elastic properties E (Young modulus) and ν (Poisson's coefficient). No loads are applied on the wall, except for those coming from the beam.

- (a) Write down the equations in the wall assuming a plane stress behavior.
- (b) Write down the equations for the beam modified because of the presence of the wall.
- (c) Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at $y = 0$
- (d) Suggest transmission conditions for u and the tangent component of the traction on the wall at $y = 0$. Discuss the implications if this component is not assumed to be zero.

Solution (a):
Hooke's Law:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\gamma_{xy} \end{bmatrix}$$

Where the strains vector is defined as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} \\ \nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ \frac{1-\nu}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \end{bmatrix}$$

Momentum equation (equilibrium):

$$\nabla\sigma + b = 0$$

$$\frac{E}{1-v^2} \begin{bmatrix} \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}) + \frac{\partial}{\partial y}(\frac{1-\nu}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})) \\ \frac{\partial}{\partial y}(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{\partial}{\partial x}(\frac{1-\nu}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})) \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Boundary conditions:
Fixed displacements on laterals and bottom sides, and the traction forces of the beam are the boundary condition on the top of the wall.

Solution(b):
Considering the governing equation:

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = f$$

The wall will give a distributed load to the beam, therefore the governing equation yields:

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = f - t\ \sigma_{yy}\big|_{y=0}$$

Where "t" corresponds to the thickness of the wall.
And:

$$\sigma_{yy}\big|_{y=0} = \frac{E}{1-\nu^2}(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\Big|_{y=0}$$

Solution (c):
Knowing that the interface Γ is between the wall and the beam, we know the vertical displacements must be the same for the wall and beam all over the Γ. Therefore its jump across the interface must be equal to zero.
Therefore the first transmission condition yields:

$$[\![v]\!]_\Gamma = 0$$

Since the reaction of the wall must be equal to the imposed force coming from the beam, the normal traction force must be the same on Γ, hence, the second transmission condition yields:

$$[\![n \cdot (\nabla \cdot \sigma)]\!]_\Gamma = 0$$

Solution (d):
The completely fulfill the Euler-Bernoulli theory, the displacements for the beam and the wall must be set equal to zero everywhere, meanwhile for the traction forces must be set to zero on Γ for both the wall and the beam.
If these conditions are not set to zero, the angular momentum on the linear elastic solid will be unbalanced.

3.2 Problem 2

Let $\mathcal{S}_D$ and $\mathcal{S}_S$ be the Dirichlet-to-Neumann operators for the Darcy and the Stokes problems, respectively (see the class notes, chapter 3). The Steklov-Poincaré equation can be written as

$$\mathcal{S}_S(\lambda) = \mathcal{S}_D(\lambda)$$

where λ is the normal velocity on Γ, the interface between the Darcy and the Stokes regions.

- (a) Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

- (b) Write down the matrix form of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems.
- (c) Identify the Richardson iteration for the algebraic problem in (a) resulting from (b).

Solution (a):

Sub-domain Ω_S :

$$\begin{aligned} -\nu\Delta u_S + \nabla p_S = f &\longrightarrow in \;\; \Omega_S \\ \nabla \cdot u_S = 0 &\longrightarrow in \;\; \Omega_S \\ u_S = \bar{u}_S &\longrightarrow on \;\; \Gamma_S \end{aligned}$$

Sub-domain Ω_D :

$$\begin{aligned} k^{-1}u_D + \nabla\phi = 0 &\longrightarrow in \;\; \Omega_D \\ \nabla \cdot u_D = 0 &\longrightarrow in \;\; \Omega_D \\ n \cdot u_D = \bar{u}_{n,D} &\longrightarrow on \;\; \Gamma_D \end{aligned}$$

Interface conditions:

$$n \cdot u_S = n \cdot u_D$$

$$p_S - (n \cdot \nu\nabla u_S) \cdot n = \phi$$

$$u_S \cdot t = -\frac{\sqrt{k}}{\alpha_{BJ}}(n \cdot \nu\nabla u_S) \cdot t$$

Where t is a unit tangential vector on Γ.

- Stokes weak form:

$$-\int_{\Omega_S} \delta u_S \cdot \nu\Delta u_S + \int_{\Omega_S} \delta u_S \cdot \nabla p_S = \int_{\Omega_S} \delta u_S \cdot f$$

$$\int_{\Omega_S} w_S \left(\nabla \cdot u_S\right) = 0$$

Integrating by parts and setting $\delta\boldsymbol{u}_S = \mathbf{0}$ on Γ_S yields:

$$\int_{\Omega_S} \nabla\delta u_S : \nu\nabla u_S - \int_{\Omega_S} p_S \left(\nabla \cdot \delta u_S\right) - \int_{\Gamma} \delta u_S \cdot \left[n_S \cdot \left(-p_S I + \nu\nabla u_S\right)\right] = \int_{\Omega_S} \delta u_S \cdot f$$

$$\int_{\Omega_S} w_S \left(\nabla \cdot u_S\right) = 0$$

Where δu_S is a vector test function and w_S is a scalar test function.

- Darcy weak form:

$$\int_{\Omega_D} \delta\boldsymbol{u}_D \cdot k^{-1}\boldsymbol{u}_D + \int_{\Omega_D} \delta\boldsymbol{u}_D \cdot \boldsymbol{\nabla}\phi = 0$$

$$\int_{\Omega_D} w_D \left(\boldsymbol{\nabla} \cdot \boldsymbol{u}_D\right) = 0$$

Integrating by parts and setting $\delta\boldsymbol{u}_D = \mathbf{0}$ on Γ_D yields

$$\int_{\Omega_D} \delta\boldsymbol{u}_D \cdot k^{-1}\boldsymbol{u}_D - \int_{\Omega_D} \phi \left(\boldsymbol{\nabla} \cdot \delta\boldsymbol{u}_D\right) + \int_{\Gamma} \delta\boldsymbol{u}_D \cdot \phi\boldsymbol{n}_D = 0$$

$$\int_{\Omega_D} w_D \left(\boldsymbol{\nabla} \cdot \boldsymbol{u}_D\right) = 0$$

Where δu_D is a vector test function and w_D is a scalar test function.
After a Galerkin discretization of the problem, the matrix form yields:
Stoke's problem:

$$\begin{bmatrix} K_S & G_S \\ G_S^T & 0 \end{bmatrix} \begin{bmatrix} U_S \\ P_S \end{bmatrix} = \begin{bmatrix} f_S \\ h_S \end{bmatrix}$$

Darcy's problem:

$$\begin{bmatrix} M_D & G_D \\ G_D^T & 0 \end{bmatrix} \begin{bmatrix} U_D \\ \Phi_D \end{bmatrix} = \begin{bmatrix} f_D \\ h_D \end{bmatrix}$$

Where $U = [U_I^T, \lambda^T]^T$. Therefore, the system of equations can be rewritten as:

Sub-domain Ω_S :

$$\begin{bmatrix} A_{II}^S & A_{I\Gamma}^S & B_{II}^S \\ A_{\Gamma I}^S & A_{\Gamma\Gamma}^S & B_{\Gamma I}^S \\ B_{II}^S & B_{I\Gamma}^S & 0 \end{bmatrix} \begin{bmatrix} U_I^S \\ \lambda \\ P^S \end{bmatrix} = \begin{bmatrix} f_{II}^S \\ f_{I\Gamma}^S \\ h_I^S \end{bmatrix}$$

Sub-domain Ω_D :

$$\begin{bmatrix} A_{II}^D & A_{I\Gamma}^D & B_{II}^D \\ A_{\Gamma I}^D & A_{\Gamma\Gamma}^D & B_{\Gamma I}^D \\ B_{II}^D & B_{I\Gamma}^D & 0 \end{bmatrix} \begin{bmatrix} U_I^D \\ \lambda \\ \Phi^D \end{bmatrix} = \begin{bmatrix} f_{II}^D \\ f_{I\Gamma}^D \\ h_I^D \end{bmatrix}$$

Combining both systems, we obtain:

Domain $\Omega_S \cup \Omega_D$:

$$\begin{bmatrix} A_{II}^S & B_{II}^S & A_{I\Gamma}^S & 0 & 0 \\ B_{II}^S & 0 & B_{\Gamma I}^S & 0 & 0 \\ A_{\Gamma I}^S & B_{\Gamma I}^S & A_{\Gamma\Gamma}^S + A_{\Gamma\Gamma}^D & A_{\Gamma I}^D & B_{\Gamma I}^D \\ 0 & 0 & A_{I\Gamma}^D & A_{II}^D & B_{II}^D \\ 0 & 0 & B_{I\Gamma}^D & B_{II}^D & 0 \end{bmatrix} \begin{bmatrix} U_I^S \\ P^S \\ \lambda \\ U_I^D \\ \Phi_I^D \end{bmatrix} = \begin{bmatrix} f_{II}^S \\ h_I^S \\ f_{I\Gamma}^S + f_{I\Gamma}^D \\ f_{II}^D \\ h_I^D \end{bmatrix}$$

By combining all the degrees of freedom of velocity and pressure in each subdomain as $\mathcal{U}_S = \left[\boldsymbol{U}_S^{int^T}, \boldsymbol{P}_S^T\right]^T$ and $\mathcal{U}_D = \left[\boldsymbol{U}_D^{int^T}, \boldsymbol{\Phi}_D^T\right]^T$, the matrix form is further simplified to:

$$\begin{bmatrix} \mathcal{A}_{SS} & \mathcal{A}_{S\Gamma} & 0 \\ \mathcal{A}_{\Gamma S} & \mathcal{A}_{\Gamma\Gamma} & \mathcal{A}_{\Gamma D} \\ 0 & \mathcal{A}_{D\Gamma} & \mathcal{A}_{DD} \end{bmatrix} \begin{Bmatrix} \mathcal{U}_S \\ \lambda \\ \mathcal{U}_D \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_S \\ \boldsymbol{F}_\Gamma \\ \boldsymbol{F}_D \end{Bmatrix}$$

The first equation gives:

$$\mathcal{U}_S = \mathcal{A}_{SS}^{-1} \left(\boldsymbol{F}_S - \mathcal{A}_{S\Gamma} \boldsymbol{\lambda}\right)$$

The third equation gives:

$$\mathcal{U}_D = \mathcal{A}_{DD}^{-1} \left(F_D - \mathcal{A}_{D\Gamma} \lambda\right)$$

And the second equations gives:

$$\mathcal{A}_{\Gamma S} \mathcal{U}_S + \mathcal{A}_{\Gamma\Gamma} \lambda + \mathcal{A}_{\Gamma D} \mathcal{U}_D = F_\Gamma$$

Eventually, after substituting the first equation and the third equation into the second equation the following equation is obtained:

$$\left(\mathcal{A}_{\Gamma\Gamma} - \mathcal{A}_{\Gamma S} \mathcal{A}_{SS}^{-1} \mathcal{A}_{S\Gamma} - \mathcal{A}_{\Gamma D} \mathcal{A}_{DD}^{-1} \mathcal{A}_{D\Gamma}\right) \lambda = F_\Gamma - \mathcal{A}_{\Gamma S} \mathcal{A}_{SS}^{-1} \boldsymbol{F}_1 - \mathcal{A}_{\Gamma D} \mathcal{A}_{DD}^{-1} F_2$$

Which is written as:

$$(\mathcal{S}_S - \mathcal{S}_D)\,\boldsymbol{\lambda} = \boldsymbol{G}$$

Where:

$$\mathcal{S}_S = A^{(S)}_{\Gamma\Gamma} - \mathcal{A}_{\Gamma S}\mathcal{A}^{-1}_{SS}\mathcal{A}_{S\Gamma}$$
$$\mathcal{S}_D = \mathcal{A}_{\Gamma D}\mathcal{A}^{-1}_{DD}\mathcal{A}_{D\Gamma} - A^{(D)}_{\Gamma\Gamma}$$
$$G = F_\Gamma - \mathcal{A}_{\Gamma S}\mathcal{A}^{-1}_{SS}F_1 - \mathcal{A}_{\Gamma D}\mathcal{A}^{-1}_{DD}F_2$$

For the problem at hand, G is given to be 0 in the question.

Solution (b):
The Dirichlet-Neumann scheme yields:

Sub-domain Ω_S :

$$-\nu\Delta u_S^{[k]} + \nabla p_S^{[k]} \longrightarrow in \;\; \Omega_S$$
$$\nabla \cdot u_S^{[k]} = 0 \longrightarrow in \;\; \Omega_S$$
$$u_S^{[k]} = \bar{u}_S \longrightarrow on \;\; \Gamma_S$$
$$n \cdot u_S^{[k]} = n \cdot u_D^{[k-1]} \longrightarrow on \;\; \Gamma$$
$$u_S^{[k]} \cdot t = -\frac{\sqrt{k}}{\alpha_{BJ}}(n \cdot \nu\nabla u_S^{[k]}) \cdot t \longrightarrow on \;\; \Gamma$$

Sub-domain Ω_D :

$$k^{-1}u_D^{[k]} + \nabla\phi^{[k]} = 0 \longrightarrow in \;\; \Omega_D$$
$$\nabla \cdot u_D^{[k]} \longrightarrow in \;\; \Omega_D$$
$$n \cdot u_D^{[k]} = \bar{u}_{n,D} \longrightarrow on \;\; \Gamma_D$$
$$\phi^{[k]} = p_S^{[][l]} - (n \cdot \nu\nabla u_S^{[l]}) \cdot n \longrightarrow on \;\; \Gamma$$

The matrix form of the scheme yields:

Domain Ω_S :

$$\begin{bmatrix} A^S_{II} & B^S_{II} & A_{I\Gamma} \\ B^S_{II} & 0 & B^S_{\Gamma I} \\ A^S_{\Gamma I} & B^S_{I\Gamma} & A^S_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_I^{S\ [k]} \\ P^{S\ [k]} \\ \lambda^{[k]} \end{bmatrix} = \begin{bmatrix} f^S_{II} \\ h^S_I \\ f^S_{I\Gamma} - A^D_{\Gamma\Gamma}\lambda^{[k-1]} - A^D_{\Gamma I}U_I^{D\ [k-1]} \end{bmatrix}$$

Domain Ω_D :

$$\begin{bmatrix} A^D_{II} & B^D_{II} \\ B^D_{II} & 0 \end{bmatrix} \begin{bmatrix} U_I^{S\ [k]} \\ \Phi_I^{D\ [k]} \end{bmatrix} = \begin{bmatrix} f^D_{II} - A_{I\Gamma}\lambda^{[l]} \\ h^D_I \end{bmatrix}$$

Solution (c):
Richardson scheme:

Domain Ω_S :

$$\begin{bmatrix} U_I^{S\ [k]} \\ P^{S\ [k]} \\ \lambda^{[k]} \end{bmatrix} = \begin{bmatrix} U_I^{S\ [k-1]} \\ P^{S\ [k-1]} \\ \lambda^{[k-1]} \end{bmatrix} + \left(\begin{bmatrix} f_{II}^S \\ h_I^S \\ f_{I\Gamma}^S - A_{\Gamma\Gamma}^D \lambda^{[k-1]} - A_{\Gamma I}^D U_I^{D\ [k-1]} \end{bmatrix} - \begin{bmatrix} A_{II}^S & B_{II}^S & A_{I\Gamma} \\ B_{II}^S & 0 & B_{\Gamma I}^S \\ A_{\Gamma I}^S & B_{I\Gamma}^S & A_{\Gamma\Gamma}^S \end{bmatrix} \begin{bmatrix} U_I^{S\ [k-1]} \\ P^{S\ [k-1]} \\ \lambda^{[k-1]} \end{bmatrix} \right)$$

Domain Ω_D :

$$\begin{bmatrix} U_I^{S\ [k]} \\ \Phi_I^{D\ [k]} \end{bmatrix} = \begin{bmatrix} U_I^{S\ [k-1]} \\ \Phi_I^{D\ [k-1]} \end{bmatrix} + \left(\begin{bmatrix} f_{II}^D - A_{I\Gamma}\lambda^{[l]} \\ h_I^D \end{bmatrix} - \begin{bmatrix} A_{II}^D & B_{II}^D \\ B_{II}^D & 0 \end{bmatrix} \begin{bmatrix} U_I^{S\ [k-1]} \\ \Phi_I^{D\ [k-1]} \end{bmatrix} \right)$$

4 Monolithic and partitioned schemes in time

Consider the one-dimensional, transient, heat transfer equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= f \qquad \text{in } [0,1] \\ u(x=0,t) &= 0 \\ u(x=1,t) &= 0 \\ u(x,t=0) &= 0 \end{aligned}$$

4.1 Problem 1

Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\kappa = 1, f = 1, \delta t = 1$
Solution: The weak form of the problem is obtained by multiplying by a test function v and integrating over the domain Ω:

$$\int_\Omega v \frac{\partial u}{\partial t} d\Omega - \int_\Omega v\kappa \frac{\partial^2 u}{\partial x^2} d\Omega = \int_\Omega v f d\Omega$$

Integrating by parts yields:

$$\int_\Omega v \frac{\partial u}{\partial t} d\Omega + \int_\Omega \frac{\partial v}{\partial x} \kappa \frac{\partial u}{\partial x} d\Omega - \int_{\partial\Omega} v\kappa \frac{\partial u}{\partial x} n d\Gamma = \int_\Omega v f d\Omega$$

The boundary condition term correspond to a Neumann condition. Since we only have Dirichlet condition, this term vanishes and the weak form yields:

$$\int_\Omega v \frac{\partial u}{\partial t} d\Omega + \int_\Omega \frac{\partial v}{\partial x} \kappa \frac{\partial u}{\partial x} d\Omega = \int_\Omega v f d\Omega$$

Using Galerkin finite elements the system yields:

$$M = \int_0^1 N^T N d\Omega$$

$$K = \int_0^1 \kappa \frac{\partial N^T}{\partial x} \frac{\partial N^T}{\partial x} d\Omega$$

$$F = \int_0^1 N^T f d\Omega$$

The matrix form of the system yields as follows:

$$M\frac{du}{dt} + Ku = F$$

The BDF1 time discretization scheme is defined as follows:

$$M\frac{u^{n+1} - u^n}{\delta t} + Ku^{n+1} = F^{n+1}$$

Considering the proposed values ($\delta t = 1 \leftrightarrow f = 1$) the system yields:

$$(M + K)u^{n+1} = F + Mu^n$$

$$u^{n+1} = (M + K)^{-1}(F + Mu^n)$$

4.2 Problem 2

Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements ($h = 0.2$), while the right subdomain is composed of 3 elements ($h = 0.2$). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as $u_0, u_1, u_2, u_3, u_4, u_5$ The interface is at u_2

Solution:

Splitting the domain into two sub-domains ($\Omega_1 = [0, 0.4]$ and $\Omega_1 = [0.4, 1]$), where the interface (Γ) lies on u_2 node. The weak for this problem yields:

Sub-domain Ω_1:

$$\int_{\Omega_1} v\frac{\partial u}{\partial t} d\Omega_1 + \int_{\Omega_1} \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x} d\Omega_1 - \int_{\Gamma} v\kappa\frac{\partial u}{\partial x} n_1 d\Gamma = \int_{\Omega_1} vf d\Omega_1$$

Sub-domain Ω_2:

$$\int_{\Omega_2} v\frac{\partial u}{\partial t} d\Omega_2 + \int_{\Omega_2} \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x} d\Omega_2 - \int_{\Gamma} v\kappa\frac{\partial u}{\partial x} n_2 d\Gamma = \int_{\Omega_2} vf d\Omega_2$$

Recalling the transmission conditions:

$$[\![u]\!]_\Gamma = 0 \longrightarrow \text{1st transmission condition}$$

$$[\![\kappa\frac{\partial u}{\partial x} n]\!]_\Gamma = 0 \longrightarrow \text{2nd transmission condition}$$

The two equations must be summed to solve the problem in a monolithic way, which yields:

$$\underbrace{\int_{\Omega_1} v\frac{\partial u}{\partial t}d\Omega_1 + \int_{\Omega_2} v\frac{\partial u}{\partial t}d\Omega_2}_{\int_\Omega v\frac{\partial u}{\partial t}d\Omega} + \underbrace{\int_{\Omega_1} \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x}d\Omega_1 + \int_{\Omega_2} \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x}d\Omega_2}_{\int_\Omega \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x}d\Omega} ...$$

$$+ \underbrace{\left(-\int_\Gamma v\kappa\frac{\partial u}{\partial x}n_1 d\Gamma - \int_\Gamma v\kappa\frac{\partial u}{\partial x}n_2 d\Gamma\right)}_{0} = \underbrace{\int_{\Omega_1} vf d\Omega_1 + \int_{\Omega_2} vf d\Omega_2}_{\int_\Omega vf d\Omega}$$

Therefore:

$$\int_\Omega v\frac{\partial u}{\partial t}d\Omega + \int_\Omega \frac{\partial v}{\partial x}\kappa\frac{\partial u}{\partial x}d\Omega = \int_\Omega vf d\Omega$$

4.3 Problem 3

Obtain the algebraic form of the Dirichlet-to-Neumann operator (Steklov-Poincaré's operator) for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}

Solution:

Recalling the system of equations for the first part of the current problem:

$$u^{n+1} = (M+K)^{-1}(F+Mu^n) \longrightarrow (M+K)u^{n+1} = (F+Mu^n)$$

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I^{(1)\ [n+1]} \\ u_\Gamma^{[n+1]} \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ f_\Gamma^{(1)} \end{bmatrix} + \begin{bmatrix} M_{II}^{(1)} & M_{I\Gamma}^{(1)} \\ M_{\Gamma I}^{(1)} & M_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{[n]} \\ u_\Gamma^{[n]} \end{bmatrix}$$

Note that the only unknown we have in this system is $u_I^{(1)\ [n+1]}$

$$A_{II}^{(1)}u_I^{(1)\ [n+1]} + A_{I\Gamma}^{(1)}u_\Gamma^{[n+1]} = f_I^{(1)} + M_{II}^{(1)}u_I^{(1)\ [n]} + M_{I\Gamma}^{(1)}u_\Gamma^{[n]}$$

$$u_I^{(1)\ [n+1]} = A_{II}^{(1)\ -1}\left(f_I^{(1)} + M_{II}^{(1)}u_I^{(1)\ [n]} + M_{I\Gamma}^{(1)}u_\Gamma^{[n]} - A_{I\Gamma}^{(1)}u_\Gamma^{[n+1]}\right)$$

Alternate solution:

Recalling the system of equations for the first part of the current problem:

$$U^{[n+1]} = (M+K)^{-1}(F+MU^{[n]}) \longrightarrow \underbrace{(M+K)}_{A}U^{[n+1]} = \underbrace{(F+Mu^{[n]})}_{B}$$

Therefore:

$$U^{[n+1]} = A^{-1}B$$

The complete matrix form for this specific problem yields:

$$\begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & 0 \\ A_{10} & A_{11} & A_{12} & 0 & 0 & 0 \\ 0 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & A_{33} & A_{34} & 0 \\ 0 & 0 & 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} U_0^{[n+1]} \\ U_1^{[n+1]} \\ U_2^{[n+1]} \\ U_3^{[n+1]} \\ U_4^{[n+1]} \\ U_5^{[n+1]} \end{bmatrix} = \begin{bmatrix} B_0^{[n]} \\ B_1^{[n]} \\ B_2^{[n]} \\ B_3^{[n]} \\ B_4^{[n]} \\ B_5^{[n]} \end{bmatrix}$$

If we know from Dirichlet boundary conditions that $u_0 = u_5 = 0$, the system reduces to:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_1^{[n+1]} \\ U_2^{[n+1]} \\ U_3^{[n+1]} \\ U_4^{[n+1]} \end{bmatrix} = \begin{bmatrix} B_1^{[n]} \\ B_2^{[n]} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

Where:

- Sub-domain Ω_1:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}^{\Omega_1} \end{bmatrix} \begin{bmatrix} U_1^{[n+1]} \\ U_2^{[n+1]} \end{bmatrix} = \begin{bmatrix} B_1^{[n]} \\ B_2^{[n]} \end{bmatrix}$$

Note that the only unknown we have in this system is $U_I^{(1)\ [n+1]}$:

$$A_{11}U_1^{[n+1]} = B_1^{[n]} - A_{12}U_2^{[n+1]}$$

Where:

$$-A_{12}U_2^{[n+1]} \longrightarrow \text{Dirichlet boundary condition}$$

4.4 Problem 4

Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right sub-domain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = \kappa \partial_x u^{n+1}$ at the coordinate of node 2.

Solution:

From the reduced system:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_1^{[n+1]} \\ U_2^{[n+1]} \\ U_3^{[n+1]} \\ U_4^{[n+1]} \end{bmatrix} = \begin{bmatrix} B_1^{[n]} \\ B_2^{[n]} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

We have that:

- Sub-domain Ω_2:

$$\begin{bmatrix} A_{22}^{\Omega_2} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{[n+1]} \\ U_3^{[n+1]} \\ U_4^{[n+1]} \end{bmatrix} = \begin{bmatrix} B_2^{[n]} - A_{21}U_1^{[n+1]} - A_{22}^{\Omega_1}U_2^{[n+1]} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

Where:

$$-A_{21}U_1^{[n+1]} - A_{22}^{\Omega_1}U_2^{[n+1]} \longrightarrow \text{Neumann boundary condition}$$

4.5 Problem 5

Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.
Solution:
Staggered approach: Fist we need to define a prediction $\tilde{U}^{n+1}$ to replace all the unknowns on the RHS of the equation, allowing parallel computing making it faster:

$$\underbrace{\tilde{U}^{n+1} = U^n}_{\text{Fist order approximation}}$$

$$\underbrace{\tilde{U}^{n+1} = 2U^n - U^{n-1}}_{\text{Second order approximation}}$$

The iterative scheme will be performed for each time step and iterating over k until convergence has reached:

- Sub-domain Ω_2:

$$\begin{bmatrix} A_{22}^{\Omega_2} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{[n+1](k)} \\ U_3^{[n+1](k)} \\ U_4^{[n+1](k)} \end{bmatrix} = \begin{bmatrix} B_2^{[n]} - A_{21}\tilde{U}_1^{[n+1](k-1)} - A_{22}^{\Omega_1}\tilde{U}_2^{[n+1](k-1)} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

- Sub-domain Ω_1:

$$A_{11}U_1^{[n+1](k)} = B_1^{[n]} - A_{12}\tilde{U}_2^{[n+1](k)}$$

Convergence or stability of this scheme is not guaranteed.

4.6 Problem 6

Do the same for a substitution and an iteration by subdomains scheme.
Solution:
Substitution:
The idea is now only predict the unknown in one subdomain and using the resolved

variable for the solution of the the other subdomain. Notice that this scheme will not be parallelizable, but will have higher accuracy that staggered approach, and it is defined as follows:

- Sub-domain Ω_2:

$$\begin{bmatrix} A_{22}^{\Omega_2} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{[n+1](k)} \\ U_3^{[n+1](k)} \\ U_4^{[n+1](k)} \end{bmatrix} = \begin{bmatrix} B_2^{[n]} - A_{21}\tilde{U}_1^{[n+1](k-1)} - A_{22}^{\Omega_1}\tilde{U}_2^{[n+1](k-1)} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

- Sub-domain Ω_1:

$$A_{11}U_1^{[n+1](k)} = B_1^{[n]} - A_{12}U_2^{[n+1](k)}$$

Convergence or stability of this scheme is not guaranteed.
Iteration bu subdomains:
Now the problem can be iterated without any predictions. If we reach convergence, we recover the solution of the monolithic problem, and yields as follows:

- Sub-domain Ω_2:

$$\begin{bmatrix} A_{22}^{\Omega_2} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{[n+1](k)} \\ U_3^{[n+1](k)} \\ U_4^{[n+1](k)} \end{bmatrix} = \begin{bmatrix} B_2^{[n]} - A_{21}U_1^{[n+1](k-1)} - A_{22}^{\Omega_1}U_2^{[n+1](k-1)} \\ B_3^{[n]} \\ B_4^{[n]} \end{bmatrix}$$

- Sub-domain Ω_1:

$$A_{11}U_1^{[n+1](k)} = B_1^{[n]} - A_{12}U_2^{[n+1](k)}$$

Convergence or stability of this scheme is not guaranteed.

4.7 Problem 7

Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface), using Nitsche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

Solution:
Recalling the weak form for the left subdomain Ω_1:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_{\partial\Omega_1} = (v, f)_{\Omega_1}$$

Adding the Nitsche's method terms as a better condition version of the penalthy method:

$$\begin{aligned}\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \left\langle v, \kappa\frac{\partial u}{\partial x}n_1\right\rangle_{\partial\Omega_1} + \alpha\frac{\kappa}{h}\langle v, u\rangle_{\partial\Omega_1} - \kappa\left\langle \frac{\partial v}{\partial x}n_1, u\right\rangle_{\partial\Omega_1} &= (v, f)_{\Omega_1} \\ &+ \alpha\frac{\kappa}{h}\langle v, \bar{u}\rangle_{\partial\Omega_1} - \kappa\left\langle \frac{\partial v}{\partial x}n_1, \bar{u}\right\rangle_{\partial\Omega_1}\end{aligned}$$

Where:

- $\alpha \longrightarrow$ penalty parameter to ensure stability
- $h \longrightarrow$ element size
- $\bar{u} \longrightarrow$ prescribed Dirichlet function

For this case the left subdomain $\Omega_1 = [0, 0.4]$, therefore it yields:

$$\begin{aligned}&\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \kappa\left[\left(v\frac{\partial u}{\partial x}n_1\right)\Big|_{x=0.4} - \left(v\frac{\partial u}{\partial x}n_1\right)\Big|_{x=0}\right] + \alpha\frac{\kappa}{h}\left[(vu)|_{x=0.4} - (vu)|_{x=0}\right] \\ &\quad - \kappa\left[\left(\frac{\partial v}{\partial x}n_1 u\right)\Big|_{x=0.4} - \left(\frac{\partial v}{\partial x}n_1 u\right)\Big|_{x=0}\right] = (v, f)_{\Omega_1} + \alpha\frac{\kappa}{h}\left[(v\bar{u})|_{x=0.4} - (v\bar{u})|_{x=0}\right] \\ &\quad - \kappa\left[\left(\frac{\partial v}{\partial x}n_1 \bar{u}\right)\Big|_{x=0.4} - \left(\frac{\partial v}{\partial x}n_1 \bar{u}\right)\Big|_{x=0}\right]\end{aligned}$$

Considering $n_1 = 1$ at $x = 0.4$ and $n_1 = -1$ at $x = 0$:

$$\begin{aligned}&\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \kappa\frac{\partial u}{\partial x}\right)_{\Omega_1} - \kappa\left[v\frac{\partial u}{\partial x}|_{x=0.4} + v\frac{\partial u}{\partial x}|_{x=0}\right] + \alpha\frac{\kappa}{h}[vu|_{x=0.4} - vu|_{x=0}] \\ &\quad - \kappa\left[\frac{\partial v}{\partial x}u|_{x=0.4} + \frac{\partial v}{\partial x}u|_{x=0}\right] = (v, f)_{\Omega_1} + \alpha\frac{\kappa}{h}[v\bar{u}|_{x=0.4} - v\bar{u}|_{x=0}] \\ &\quad - \kappa\left[\frac{\partial v}{\partial x}\bar{u}|_{x=0.4} + \frac{\partial v}{\partial x}\bar{u}|_{x=0}\right]\end{aligned}$$

Recalling the Finite Element approximation derived in problem 1 of this section, obtain the following discrete problem:

$$\begin{aligned}&\int_0^{0.4} N^T N d\Omega\frac{dU}{dt} + \int_0^{0.4}\kappa\frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x}d\Omega U - \kappa\left[N^T\frac{\partial N_2}{\partial x}U_2 + N^T\frac{\partial N_0}{\partial x}U_0\right] + \alpha\frac{\kappa}{h}\left[N^T U_2 - N^T U_0\right] \\ &\quad - \kappa\left[\frac{\partial N^T}{\partial x}U_2 + \frac{\partial N^T}{\partial x}U_0\right] = \int_0^{0.4} N^T f d\Omega + \alpha\frac{\kappa}{h}\left[N^T U_2^{(\Omega_2)} - N^T(0)\right] \\ &\quad - \kappa\left[\frac{\partial N^T}{\partial x}U_2^{(\Omega_2)} + \frac{\partial N^T}{\partial x}(0)\right]\end{aligned}$$

Where:

- $N = [N_0, N_1, N_2]$
- $U = [U_0, U_1, U_2]$
- $U_2^{(\Omega_2)} \longrightarrow$ Dirichlet value imposed at the interface, where Ω_2 super index indicates that this value is computed by solving the problem in sub-domain Ω_2.

To simplify this expression we consider the following:

- M is the mass matrix
- K is the stiffness matrix
- $C = N^T$
- $D = \frac{\partial N^T}{\partial x}$
- $\frac{\partial N_2}{\partial x} = \frac{\partial N_0}{\partial x} = \frac{h}{2} \longrightarrow$ due to the use of linear elements of equal size

The system yields as follows:

$$M\frac{dU}{dt}+KU-\kappa\frac{h}{2}C\left(U_2+U_0\right)+\alpha\frac{\kappa}{h}C\left(U_2-U_0\right)-\kappa D\left(U_2+U_0\right) = F+\alpha\frac{\kappa}{h}CU_2^{(\Omega_2)}-\kappa DU_2^{(\Omega_2)}$$

Further simplification yields:

$$M\frac{dU}{dt}+KU-\kappa\left(\frac{h}{2}C-\frac{\alpha}{h}+D\right)U_2-\kappa\left(\frac{h}{2}C+\frac{\alpha}{h}+D\right)U_0 = F+\left(\alpha\frac{\kappa}{h}C-\kappa D\right)U_2^{(\Omega_2)}$$

Using BDF1 time discretization, the system of equations is written as:

$$\begin{aligned} M\frac{U^{n+1}-U^n}{\delta t} + KU^{n+1} - \kappa\left(\frac{h}{2}C-\frac{\alpha}{h}+D\right)U_2^{n+1} - \kappa\left(\frac{h}{2}C+\frac{\alpha}{h}+D\right)U_0^{n+1} \\ = F + \left(\alpha\frac{\kappa}{h}C-\kappa D\right)U_2^{(n+1)(\Omega_2)} \end{aligned}$$

The condition number of the traditional methods increases. On the other hand, the condition number of the Nitsche's method stays bounded for fixed h. Of this reason the traditional methods may cause troubles for iterative solvers such as multigrid method.
It is now sufficient to take: $\alpha > 2c_i$ to ensure stability.
c_i depends on the shape of the elements, so for non-stretched elements $c_i = \mathcal{O}(1)$. For penalty method, the required value for α is difficult to estimate. In practice it is taken very large (10^6), which can result in ill-conditioned systems of equations, meanwhile for Nitsche's method lower values of α can be used and thus, better conditioned systems.

5 Operator splitting techniques

Consider the one dimensional, transient, convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} + a_x\frac{\partial u}{\partial x} = f \quad \text{in } [0,1]$$
$$u(x=0,t)=0$$
$$u(x=1,t)=0$$
$$u(x,t=0)=0$$

with $\kappa = 1, a_x = 1, f = 1$

5.1 Problem 1

Discretize it in space using finite elements (3 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size δt

Solution:

We first have to obtain the weak form of the problem by multiplying by a test function and integration over the domain:

$$\int_\Omega \delta u\frac{\partial u}{\partial t} - \int_\Omega \delta u\kappa\frac{\partial^2 u}{\partial x^2} + \int_\Omega \delta u a_x\frac{\partial u}{\partial x} = \int_\Omega \delta u f$$

Integrating by parts and neglecting the boundary term due to lack of Neumann boundary conditions it yields:

$$\int_\Omega \delta u\frac{\partial u}{\partial t} + \int_\Omega \kappa\frac{\partial \delta u}{\partial x}\frac{\partial u}{\partial x} + \int_\Omega a_x\delta u\frac{\partial u}{\partial x} = \int_\Omega \delta u f$$

Using a Galerkin discretization for space, and considering $\kappa = 1$, $a_x = 1$ and $f = 1$, it yields:

$$\int_\Omega N^T N d\Omega \;\; \frac{\partial U}{\partial t} + \int_\Omega \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x}d\Omega \;\; U + \int_\Omega N^T\frac{\partial N}{\partial x}d\Omega \;\; U = \int_\Omega N^T d\Omega$$

Where:

$$M = \int_\Omega N^T N d\Omega$$

$$K = \int_\Omega \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x}d\Omega$$

$$C = \int_\Omega N^T\frac{\partial N}{\partial x}d\Omega$$

$$F = \int_\Omega N^T d\Omega$$

Therefore the system can be simplified to:

$$M\frac{\partial U}{\partial t} + KU + CU = F$$

If we now apply a BFD1 scheme for time discretization:

$$M\frac{U^{n+1}-U^{n}}{\delta t}+KU^{n+1}+CU^{n+1}=F^{n+1}$$

Since vector F isn't time dependent, the system of equations yields:

$$(\frac{1}{\delta t}M+K+C)U^{n+1}=F+\frac{1}{\delta t}MU^{n}$$

If we want to solve 1 time step for a mesh of 3 elements and 4 nodes, we first need to apply the boundary conditions in order to reduce the system to:

$$\int_\Omega\left(\frac{1}{\delta t}\begin{bmatrix} N_2N_2 & N_2N_3 \\ N_3N_2 & N_3N_3\end{bmatrix}+\begin{bmatrix}\frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}\end{bmatrix}+\begin{bmatrix} N_2\frac{\partial N_2}{\partial x} & N_2\frac{\partial N_3}{\partial x} \\ N_3\frac{\partial N_2}{\partial x} & N_3\frac{\partial N_3}{\partial x}\end{bmatrix}\right)d\Omega\ U^{n+1}=...$$

$$...\int_\Omega\begin{Bmatrix} N_2 \\ N_3\end{Bmatrix}d\Omega+\frac{1}{\delta t}\int_\Omega\begin{bmatrix} N_2N_2 & N_2N_3 \\ N_3N_2 & N_3N_3\end{bmatrix}d\Omega\ U^{n}$$

Where $U^n=0$:

$$\int_\Omega\left(\frac{1}{\delta t}\begin{bmatrix} N_2N_2 & N_2N_3 \\ N_3N_2 & N_3N_3\end{bmatrix}+\begin{bmatrix}\frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}\end{bmatrix}+\begin{bmatrix} N_2\frac{\partial N_2}{\partial x} & N_2\frac{\partial N_3}{\partial x} \\ N_3\frac{\partial N_2}{\partial x} & N_3\frac{\partial N_3}{\partial x}\end{bmatrix}\right)d\Omega\ U^{n+1}...$$

$$...=\int_\Omega\begin{Bmatrix} N_2 \\ N_3\end{Bmatrix}d\Omega$$

If we consider the following shape functions:

$$N_2=\begin{cases} 3x & 0\le x\le 1/3 \\ 2-3x & 1/3\le x\le 2/3 \\ 0 & 2/3\le x\le 1\end{cases}$$
$$\frac{\partial N_2}{\partial x}=\begin{cases} 3 & 0\le x\le 1/3 \\ -3 & 1/3\le x\le 2/3 \\ 0 & 2/3\le x\le 1\end{cases}$$
$$N_3=\begin{cases} 0 & 0\le x\le 1/3 \\ 3x-1 & 1/3\le x\le 2/3 \\ 3-3x & 2/3\le x\le 1\end{cases}$$
$$\frac{\partial N_3}{\partial x}=\begin{cases} 0 & 0\le x\le 1/3 \\ 3 & 1/3\le x\le 2/3 \\ -3 & 2/3\le x\le 1\end{cases}$$

Solving the system of equations with the help a software (CASIO FX-CP400), we obtain:

$$\begin{bmatrix} U_2 \\ U_3\end{bmatrix}=\begin{bmatrix}\frac{6\delta t(51\delta t+1)}{2943\delta t^2+324\delta t+5} \\ \frac{6\delta t(57\delta t+1)}{2943\delta t^2+324\delta t+5}\end{bmatrix}$$

Therefore:

$$U^1=\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4\end{bmatrix}=\begin{bmatrix} 0 \\ \frac{6\delta t(51\delta t+1)}{2943\delta t^2+324\delta t+5} \\ \frac{6\delta t(57\delta t+1)}{2943\delta t^2+324\delta t+5} \\ 0\end{bmatrix}$$

5.2 Problem 2

Solve the same time step by using a first order operator splitting technique.
Solution:
If we consider a first order operator splitting technique, we have to define the following:

$$\mathcal{L} = \mathcal{L}_c + \mathcal{L}_d$$

$$\mathcal{L}_c u = a\frac{\partial u}{\partial x}$$

$$\mathcal{L}_d u = -\kappa\frac{\partial^2 u}{\partial x^2}$$

Hence, the equation is rewritten as:

$$\frac{\partial u}{\partial t} + \mathcal{L}_d u + \mathcal{L}_c u = f$$

The intermediate variables are u_c and u_d.
Fist step:

$$u_c(t_n) = u^n$$

$$\frac{\partial u_c}{\partial t} + \mathcal{L}_c u_c = 0$$

Second step:

$$u_d(t_n) = u_c(t_{n+1})$$

$$\frac{\partial u_d}{\partial t} + \mathcal{L}_d u_d = f$$

We finally will obtain the solution at the next time step as:

$$u^{n+1} = u_d(t_{n+1})$$

Therefore, the system of equations needs to be divided as follows:
Fist step:

$$U_c = U^n$$

$$(\frac{1}{\delta t}M + C)U_c^{n+1} = \frac{1}{\delta t}MU_c^n$$

Second step:

$$U_d = U_c^{n+1}$$

$$(\frac{1}{\delta t}M + K)U_d^{n+1} = F + \frac{1}{\delta t}MU_d^n$$

We finally will obtain the solution at the next time step as:

$$U^{n+1} = U_d^{n+1}$$

Solution of the problem:
Fist step:

$$U_c^1 = (\frac{1}{\delta t}M + C)^{-1}\frac{1}{\delta t}MU_c^0$$

Applying boundary conditions for $t = 0 \longrightarrow U^0 = 0$:

$$U_c^1 = 0$$

Second step:

$$U_d^1 = (\frac{1}{\delta t}M + K)^{-1}F$$

Applying boundary conditions $U_d^t = U_c^t = 0$ and solving the problem with the help of a software (CASIO FX-CP400):

$$\begin{bmatrix} U_2^1 \\ U_3^1 \end{bmatrix} = \begin{bmatrix} \frac{6\delta t}{54\delta t+5} \\ \frac{6\delta t}{54\delta t+5} \end{bmatrix}$$

Therefore:

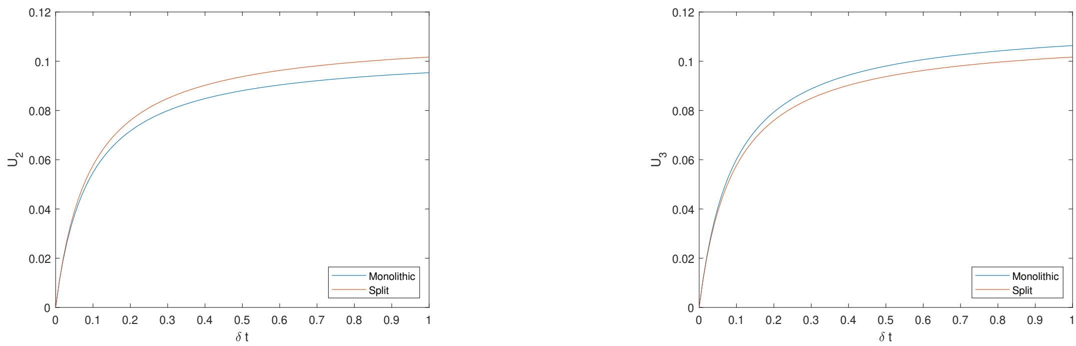
$$U^1 = \begin{bmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ U_4^1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{6\delta t}{54\delta t+5} \\ \frac{6\delta t}{54\delta t+5} \\ 0 \end{bmatrix}$$

5.3 Problem 3

Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $\delta t = 1, \delta t = 0.5, \delta t = 0.25$. Comment on the results.

Solution:

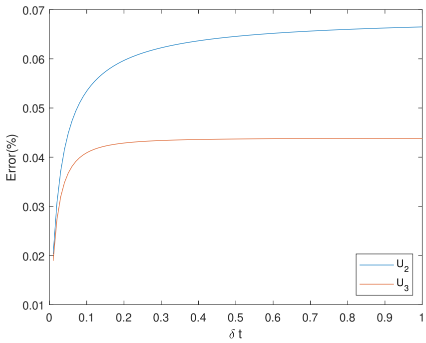
To show the behaviour of the solutions for different δt (from 0 to 1), it is worth to plot both solutions:



(a) U_2 solution for monolithic and split techniques.

(b) U_3 solution for monolithic and split techniques.

When splitting the system of equations, we are introducing an error of $\mathcal{O}(\delta t)$, therefore, it can be notice that as δt is smaller, we will reach the same solution as for the monolithic system of equations.



Error for U_2 and U_3.

6 Fractional step methods

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$\begin{aligned}
&M\tfrac{1}{\delta t}\left(\hat{U}^{n+1}-U^{n}\right)+K\hat{U}^{n+1}=f-G\tilde{P}^{n+1}\\
&DM^{-1}GP^{n+1}=\tfrac{1}{\delta t}D\hat{U}^{n+1}-DM^{-1}G\tilde{P}^{n+1}\\
&M\tfrac{1}{\delta t}\left(U^{n+1}-\hat{U}^{n+1}\right)+\alpha K\left(U^{n+1}-\hat{U}^{n+1}\right)+G\left(P^{n+1}-\tilde{P}^{n+1}\right)=0
\end{aligned}$$

6.1 Problem 1

Which is the optimal value for the α parameter?
Solution:
From [1] the incompressible Navier-Stokes equations using BDF1 yields:

$$M\frac{1}{\delta t}(U^{n+1}-U^{n})+KU^{n+1}=f-GP^{n+1}$$

$$DU^{n+1}=0$$

If we add the following equations:

$$M\frac{1}{\delta t}\left(\hat{U}^{n+1}-U^{n}\right)+K\hat{U}^{n+1}=f-G\tilde{P}^{n+1}$$

$$+$$

$$M\frac{1}{\delta t}\left(U^{n+1}-\hat{U}^{n+1}\right)+\alpha K\left(U^{n+1}-\hat{U}^{n+1}\right)+G\left(P^{n+1}-\tilde{P}^{n+1}\right)=0$$

$$=$$

$$M\frac{1}{\delta t}\left(U^{n+1}-U^{n}\right)+K\left(\hat{U}^{n+1}+\alpha U^{n+1}-\alpha\hat{U}^{n+1}\right)=f-GP^{n+1}$$

To recover the original scheme, we have to set $\alpha=1$:

$$M\frac{1}{\delta t}\left(U^{n+1}-U^{n}\right)+K\left(\cancel{\hat{U}^{n+1}}+U^{n+1}-\cancel{\hat{U}^{n+1}}\right)=f-GP^{n+1}$$

$$M\frac{1}{\delta t}\left(U^{n+1}-U^{n}\right)+KU^{n+1}=f-GP^{n+1}$$

Therefore the optimal value for the parameter is $\alpha = 1$.

6.2 Problem 2

What is the source of error of the scheme?
Solution:
Yosida method is implemented with the purpose of splitting the original problem into smaller problems, by separating the velocity field from the pressure field. The splitting of the problem always introduces an error (see previous exercise) affecting the continuity equation to stabilize the solution.
These errors can be noticed by defining the consistent incompressibility constraint:

$$DM^{-1}GP^{n+1}=DM^{-1}f-DM^{-1}KU^{n+1}+\frac{1}{\delta t}DU^{n}$$

Compared to:

$$DM^{-1}GP^{n+1}=\frac{1}{\delta t}D\hat{U}^{n+1}-DM^{-1}G\tilde{P}^{n+1}$$

We can notice that the source of errors comes form $\hat{U}$ and $\tilde{P}$.

7 ALE formulations

7.1 Problem 1

Given the spatial description of a property

$$\gamma(x,y,z,t)=\left[2x,ye^{t},z\right]$$

the equations of movement:

$$x = Xe^{t}$$
$$y = Y + e^{t} - 1$$
$$z = Z$$

and the equations of the movement of the mesh:

$$x_m = \mathcal{X} + \alpha t$$

$$y_m = \mathcal{Y} - \beta t$$
$$z_m = \mathcal{Z}$$

- (a) Obtain the description of the property in terms of the ALE coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$
- (b) Compute the velocity of the particles and the mesh velocity.

- (c) Compute the ALE description of the material temporal derivative of γ.

Solution (a):
To obtain the description of the property in terms of the ALE coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, we have to perform a substitution of the equations of the movement of the mesh into the spatial description:

$$\gamma_{ALE}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, t) = [2(\mathcal{X} + \alpha t), (\mathcal{Y} - \beta t)e^t, \mathcal{Z}]^T$$

Solution (b):
Velocity of the particles:

$$v = \frac{\partial x(X,t)}{\partial t} = [Xe^t, e^t, 0]^T$$

Velocity of the mesh:

$$v_m = \frac{\partial x(\mathcal{X},t)}{\partial t} = [\alpha, -\beta, 0]^T$$

Solution (c):
Material temporal derivative of γ:

$$\frac{d\gamma_{ALE}}{dt} = \frac{\partial \gamma_{ALE}}{\partial t} + (v - v_m) \cdot \nabla\gamma$$

Where:

- Derivative of γ_{ALE}:

$$\frac{\partial \gamma_{ALE}}{\partial t} = [2\alpha, (\mathcal{Y} - \beta(1+t))e^t, 0]^T$$

- Relative velocity:

$$v - v_m = [Xe^t, e^t, 0]^T - [\alpha, -\beta, 0]^T = [Xe^t - \alpha, e^t + \beta, 0]^T$$

- Gradient of γ:

$$\nabla\gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore substituting into the material temporal derivative of γ:

$$\frac{d\gamma_{ALE}}{dt} = \begin{bmatrix} 2\alpha \\ (\mathcal{Y} - \beta(1+t))e^t \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Xe^t - \alpha \\ e^t + \beta \\ 0 \end{bmatrix}$$

$$\frac{d\gamma_{ALE}}{dt} = \begin{bmatrix} 2\alpha \\ (\mathcal{Y} - \beta(1+t))e^t \\ 0 \end{bmatrix} + \begin{bmatrix} 2Xe^t - 2\alpha \\ e^{2t} + \beta e^t \\ 0 \end{bmatrix}$$

$$\frac{d\gamma_{ALE}}{dt} = \begin{bmatrix} 2Xe^t \\ (\mathcal{Y} - \beta t + e^t)e^t \\ 0 \end{bmatrix}$$

If we know substitute:

$$x = Xe^t = \mathcal{X} + \alpha t \longrightarrow X = (\mathcal{X} + \alpha t)e^{-t}$$

We obtain the following:

$$\frac{d\gamma_{ALE}}{dt} = \begin{bmatrix} 2(\mathcal{X} + \alpha t) \\ (\mathcal{Y} - \beta t + e^t)e^t \\ 0 \end{bmatrix}$$

7.2 Problem 2

Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?
Solution:
Navier-Stokes for incompressible flow in ALE form:

- Momentum equation:

$$\frac{\partial u_{ALE}(\mathcal{X}, t)}{\partial t} + c \cdot \nabla u(x,t) - \nabla \cdot \sigma(x,t) = \rho(x,t) b(x,t)$$

$$c = v - v_m$$

- Mass conservation:

$$\underbrace{\rho \frac{\partial u_{ALE}(\mathcal{X}, t)}{\partial t}}_{\text{Calculated at the mesh position}} + \underbrace{c \cdot \nabla \rho(x,t) + \rho(x,t) \nabla \cdot u(x,t)}_{\text{Calculated at spatial coordinates}} = 0$$

- Incompressibility:

$$\nabla \cdot u(x,t) = 0$$

Where for an incompressible flow, the Cauchy stress tensor $\sigma(x,t)$ is defined as follows:

$$\sigma(x,t) = -p(x,t)I + 2\mu \nabla^s u(x,t)$$

Therefore the momentum equation yields:

$$\underbrace{\frac{\partial u_{ALE}(\mathcal{X}, t)}{\partial t}}_{\text{Calculated at the mesh position}} + \underbrace{c \cdot \nabla u(x,t) + \nabla p(x,t) - \mu \nabla^2 u(x,t) = \rho(x,t) b(x,t)}_{\text{Calculated at spatial coordinates}}$$

For incompressible flow, the mass conservation yields:

$$\frac{\partial u_{ALE}(\mathcal{X}, t)}{\partial t} + c \cdot \nabla \rho(x,t) = 0$$

Therefore the incompressible Navier-Stokes equations in ALE form yields:

$$\underbrace{\frac{\partial u_{ALE}(\mathcal{X},t)}{\partial t} + c \cdot \nabla u(x,t) + \nabla p(x,t) - \mu \nabla^2 u(x,t) = \rho(x,t) b(x,t)}_{\text{Momentum equation}}$$

$$\underbrace{\rho \frac{\partial u_{ALE}(\mathcal{X},t)}{\partial t} + c \cdot \nabla \rho(x,t) = 0}_{\text{Mass conservation}}$$

$$\underbrace{\nabla \cdot u(x,t) = 0}_{\text{Incompressibility}}$$

For the first term (calculated at the mesh position) any time discretization with finite differences can be done, meanwhile for the other terms (calculated at spatial coordinates), the temporal derivative is evaluated as the difference from n to n+1 at a moving node.

7.3 Problem 3

Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods.
Solution:
The mesh movement must fulfill the following requirements:

- In some boundaries of the domain, it must follow the movement of the particles in the boundaries (Lagrangian boundary)
- In some boundaries of the domain, it must remain static (Eulerian boundary)
- In the interior of the domain, the mesh movement must be such that the shapes of the elements do not get excessively distorted (avoid and increase of the numerical approximation error).

The movement in the Lagrangian boundary can be prescribed a priori, or it can be the result of a couple problem computation (Fluid-structure interaction, free surface flows) [7].

There are several possibilities for computing the mesh displacements.
The boundary conditions are:

$$d_m = d_L \longrightarrow in \;\; \Gamma_{Lagrangian}$$

$$d_m = 0 \longrightarrow in \;\; \Gamma_{Eulerian}$$

In the interior of the domain, various problems can be solved. For instance:
Poisson problem:

$$-\Delta d = 0 \longrightarrow in \;\; \Omega$$

An Elasticity problem:

$$Kd = 0 \longrightarrow in \;\; \Omega$$

Different properties can be assigned to different areas of the domain. The objective is always to avoid mesh distortion, because the error of the finite element analysis is related to the shape of the elements.

Methods:

- Transfinite mapping method:

The transfinite mapping technique establishes a curvilinear coordinate system in arbitrary 2D domains. These mappings are described by adequate projectors. A projector is a linear operator which maps a true surface F onto a unit square. For example, the lofting projector φ performs a linear interpolation between two boundary curves, $\psi_1(\xi)$ and $\psi_2(\xi)$
:

$$\wp[F] = (1-\eta)\psi_1(\xi) + \eta\psi_2(\xi); \quad 0 \le \xi \le 1,\ 0 \le \eta \le 1$$

If more than 2 opposite sides of F are curvilinear, such a projector may be blended with another one of the same type in order ti interpolate a region F bounded by four curves $\psi_1(\xi)$,$\psi_2(\xi)$,$\vartheta_1(\eta)$,$\vartheta_2(\eta)$. This new projector matches exactly F on its entire boundary:

$$\begin{aligned}(\wp_1 \oplus \wp_2)[F] =& (1-\eta)\psi_1(\xi) + \eta\psi_2(\xi) + (1-\xi)\vartheta_1(\eta) + \xi\vartheta_2(\eta) - \xi\eta F(1,1) \\ & - (1-\xi)(1-\eta)F(0,0) - (1-\xi)\eta F(0,1) - \xi(1-\eta)F(1,0); \\ & 0 \le \xi \le 1,\ 0 \le \eta \le 1\end{aligned}$$

The latter may be called the transfinite bilinear Lagrange interpolant of F.
In the finite element framework, imposing discrete values to the reduced coordinates ξ and η generates very easily a mesh on surface F: e.g. equidistant reduced coordinates or values of ξ and η linked to a gradient of an unknown quantity (strain energy, local stress...) [2].

- Laplacian smoothing:

Laplacian smoothing is by far the most popular smoothing method due to its simplicity and time efficiency. Despite its long history, the original Laplacian smoothing has been presented as a heuristic method almost everywhere in the engineering literature. However, Laplacian smoothing can be derived from a finite difference approximation of the Laplace operator. In particular, it efficiently minimizes a certain convex mesh quality function with a guaranteed and unique result. Since we have found very few mentions of it minimizing a simple quadratic energy functional, we will first review the relationship of Laplacian smoothing to the gradient descent of a convex objective function, before we relate it to the popular mean ratio quality criterium and discuss suitable generalizations to polygonal and polyhedral meshes [3].

- Mesh smoothing:

In shape optimization the surface of a component is modified. If only the surface nodes are displaced and the inner nodes remain at their location, the surface layer elements will be strongly distorted. As a result, the quality of the FE calculation suffers and may no longer form a reliable basis for the optimization. To ensure realistic and high-quality results, a displacement of the surface usually necessitates a mesh displacement (mesh smoothing) in the inner of the structure in most cases.
In ALE we can always use a mesh smoothing algorithm as long as the topology of the problem is conserved.

8 Fluid-Structure Interaction

8.1 Problem 1

Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?
Solution:
The added mass effect issue appears in fluid-structure interaction problems when the densities of both the fluid and the solids are similar or close to each other, and this is of great importance to tackle. Therefore, to fix the non-convergence of the partition schemes that presents this issues, relaxation methods for the schemes works pretty well to tackle not only the high frequency eigenvectors but also the middle frequency eigenvectors by weighting the boundary conditions applied at the interface of one of the sub-domains to control the instabilities.
One relaxation method is the Aitken relaxation scheme which varies the relaxation parameter and it is defined as follows:

$$\omega_{v+1} = \frac{\Theta_i^{\Gamma\ (\nu-1)} - \Theta_i^{\Gamma\ (\nu)}}{\Theta_i^{\Gamma\ (\nu-1)} - \Theta_i^{\Gamma\ (\nu)} - \Theta_i^{\Gamma\ (\nu)} + \Theta_i^{\Gamma\ (\nu+1)}}$$

8.2 Problem 2

Consider the iteration by subdomain scheme for the heat transfer problem described in problem 1. Apply 2 iterations of the Aitken relaxation scheme to it.
Solution:
Heat transfer problem:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \qquad in[0,1]$$

$$u(x=0,t) = \bar{u}_L \tag{1}$$

$$u(x=1,t) = \bar{u}_R \tag{2}$$

$$u(x,t=0) = u_0 \tag{3}$$

Dirichlet-Neumann coupling scheme:

Sub-domain Ω_1 :

$$\begin{aligned}
&\frac{\partial u_1^{(n+1)(k)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(k)}}{\partial x^2} = f && \text{in } \Omega_1 \\
&u_1^{(n+1)(k)} = \bar{u}_L && \text{on } \Gamma_1 \\
&\kappa_1 \frac{\partial u_1^{(n+1)(k)}}{\partial n} = \kappa_2 \frac{\partial u_2^{(n+1)(k-1)}}{\partial n} && \text{on } \Gamma
\end{aligned}$$

Sub-domain Ω_2 :

$$\begin{aligned}
&\frac{\partial u_2^{(n+1)(k)}}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(k)}}{\partial x^2} = f && \text{in } \Omega_2 \\
&u_2^{(n+1)(k)} = \bar{u}_R && \text{on } \Gamma_2 \\
&u_2^{(n+1)(k)} = u_1^{(n+1)(l)} && \text{on } \Gamma
\end{aligned}$$

The Aitken relaxation scheme uses the last two iterations to approximate the relaxation parameter, therefore the first iteration will be the third one, you can simply apply a relaxation scheme for the two first iterations and then switch to Aitken relaxation scheme.

With the Gauss-Seidel scheme (not parallel), l=k, the first iteration on the Aitken scheme yields:

Sub-domain Ω_1 :

$$\begin{aligned}
&\frac{\partial u_1^{(n+1)(2)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(2)}}{\partial x^2} = f && \text{in } \Omega_1 \\
&u_1^{(n+1)(2)} = \bar{u}_L && \text{on } \Gamma_1 \\
&\kappa_1 \frac{\partial u_1^{(n+1)(2)}}{\partial n} = \kappa_2 \frac{\partial u_2^{(n+1)(1)}}{\partial n} && \text{on } \Gamma
\end{aligned}$$

Sub-domain Ω_2 :

$$\begin{aligned}
&\frac{\partial u_2^{(n+1)(2)}}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(2)}}{\partial x^2} = f && \text{in } \Omega_2 \\
&u_2^{(n+1)(2)} = \bar{u}_R && \text{on } \Gamma_2 \\
&u_2^{(n+1)(2)} = u_2^{(n+1)(1)} + w\left(u_1^{(n+1)(2)} - u_2^{(n+1)(1)}\right) && \text{on } \Gamma \\
&\text{with } w = \frac{u_2^{(n+1)(0)} - u_2^{(n+1)(1)}}{u_2^{n} - u_2^{(n+1)(1)} + u_1^{(n+1)(2)} - u_1^{(n+1)(1)}}
\end{aligned}$$

The following iteration yields as follows:

Sub-domain Ω_1 :

$$\begin{aligned}
&\frac{\partial u_1^{(n+1)(3)}}{\partial t} - \kappa_1 \frac{\partial^2 u_1^{(n+1)(3)}}{\partial x^2} = f && \text{in } \Omega_1 \\
&u_1^{(n+1)(3)} = \bar{u}_L && \text{on } \Gamma_1 \\
&\kappa_1 \frac{\partial u_1^{(n+1)(3)}}{\partial n} = \kappa_2 \frac{\partial u_2^{(n+1)(2)}}{\partial n} && \text{on } \Gamma
\end{aligned}$$

Sub-domain Ω_2 :

$$\begin{aligned}
&\frac{\partial u_2^{(n+1)}(3)}{\partial t} - \kappa_2 \frac{\partial^2 u_2^{(n+1)(3)}}{\partial x^2} = f && \text{on } \Omega_2 \\
&u_2^{(n+1)(3)} = \bar{u}_R && \text{on } \Gamma_2 \\
&u_2^{(n+1)(3)} = u_2^{(n+1)(2)} + w\left(u_1^{(n+1)(3)} - u_2^{(n+1)(2)}\right) && \text{on } \Gamma \\
&\text{with } w = \frac{u_2^{(n+1)(1)} - u_2^{(n+1)(2)}}{u_2^{(n+1)(1)} - u_2^{(n+1)(2)} + u_1^{(n+1)(3)} - u_1^{(n+1)(2)}}
\end{aligned}$$

8.3 Problem 3

Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Enforce the Dirichlet boundary conditions in $x = 0$ and $x = 1$ by using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

Solution:

Recalling the system of equations obtained in Problem 4.1:

$$\underbrace{(\frac{1}{\delta t}M + K)}_{A} U^{n+1} = \underbrace{F^{n+1} + \frac{1}{\delta t}MU^n}_{B}$$

So the system yields:

$$AU^{n+1} = B$$

Lets set up an uniform mesh of 4 elements with 5 nodes, with an $l^e = \frac{1}{4}$, and define linear isoparametric functions:

$$N_1^e(\xi) = \frac{1}{2}(1-\xi) \qquad \frac{\partial N_1^e}{\partial \xi} = \frac{-1}{2}$$

$$N_2^e(\xi) = \frac{1}{2}(1+\xi) \qquad \frac{\partial N_2^e}{\partial \xi} = \frac{1}{2}$$

Assuming $\kappa = f = 1$, the local stiffness matrices are computed as follows:

$$M^e = \int_{-1}^{1} \begin{bmatrix} N_1^e N_1^e & N_1^e N_2^e \\ N_2^e N_1^e & N_2^e N_2^e \end{bmatrix} \frac{l^e}{2} d\xi = \frac{l^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K^e = \int_{-1}^{1} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_1^e}{\partial \xi}\frac{\partial N_2^e}{\partial \xi} \\ \frac{\partial N_2^e}{\partial \xi}\frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial N_2^e}{\partial \xi} \end{bmatrix} \frac{2}{l^e} d\xi = 4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F^e = \int_{-1}^{1} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \frac{l^e}{2} d\xi = \frac{1}{8} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now since the problem was divided into 4 uniform elements, the assembly of the global stiffness matrix considering a $\delta t = 1$ yields as follows:

$$\frac{1}{24} \begin{bmatrix} 98 & -95 & 0 & 0 & 0 \\ -95 & 196 & -950 & 0 & \\ 0 & -95 & 196 & -95 & 0 \\ 0 & 0 & -95 & 196 & -95 \\ 0 & 0 & 0 & -95 & 98 \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} + 24 \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

Imposing the Dirichlet boundary conditions with Lagrangian Multipliers, the system yields as follows:

$$\frac{1}{24}\begin{bmatrix} 98 & -95 & 0 & 0 & 0 & 24 & 0 \\ -95 & 196 & -95 & 0 & 0 & 0 & 0 \\ 0 & -95 & 196 & -95 & 0 & 0 & 0 \\ 0 & 0 & -95 & 196 & -95 & 0 & 0 \\ 0 & 0 & 0 & -95 & 98 & 0 & 24 \\ 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 \end{bmatrix}\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ \bar{u}_L \\ \bar{u}_R \end{bmatrix}$$

The condition number of the resulting matrix is 38.315612859697183.

8.4 Problem 4

Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Suppose that a level set function ($\psi = 0$ at $x = 0.4$) divides the domain into a high thermal conductivity ($\kappa = 100$) subdomain ($x \in [0,0.4]$) and a low thermal conductivity ($\kappa = 1$) subdomain ($x \in (0.4, 1]$). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.

Solution:

To represent the system, we will again use the same mesh as for the previous example, noticing that the elemental mass matrix and the force vectors does not suffer any change since κ does not play a role on the computation. The stiffness matrix do suffer a change since we are dealing for a $\kappa_1 = 100$ for the first element and $\kappa_2 = 1$ for the third and fourth element, yielding the elemental stiffness matrices of this elements as follows:

$$K^1 = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^3 = K^4 = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Notice that the second element yields in the transition of the κ's change, therefore, the integral of this element will have to be split into two ($x_1 \in [0.25, 0.4]$ & $x_2 \in [0.4, 0.5]$), but since we are dealing with the isoparametric formulations, this regions are equivalent to $\xi_1 \in [-1, 0.2]$ & $\xi_2 \in [0.2, 1]$. Therefore, the elemental stiffness matrix for the second element is computed as follows:

$$K^2 = \int_{-1}^{1} \kappa \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_1^e}{\partial \xi}\frac{\partial N_2^e}{\partial \xi} \\ \frac{\partial N_2^e}{\partial \xi}\frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial N_2^e}{\partial \xi} \end{bmatrix} \frac{2}{l^e} d\xi$$

$$= 2\kappa_1 \int_{-1}^{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi + 2\kappa_2 \int_{0.2}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi$$

$$= 240\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{8}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -241.96\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The assembly of the global stiffness matrix yields as follows:

$$K = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 641.6 & -241.6 & 0 & 0 \\ 0 & -241.6 & 245.6 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

The global system using a $\delta t = 1$ yields:

$$\begin{bmatrix} \frac{4801}{12} & \frac{-9599}{24} & 0 & 0 & 0 \\ \frac{-9599}{24} & \frac{19253}{30} & \frac{-28987}{120} & 0 & 0 \\ 0 & \frac{-28987}{120} & \frac{7373}{30} & \frac{-95}{24} & 0 \\ 0 & 0 & \frac{-95}{24} & \frac{49}{6} & \frac{-95}{24} \\ 0 & 0 & 0 & \frac{-95}{24} & \frac{49}{12} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

Applying the Dirichlet boundary conditions with Lagrange Multipliers the system yields:

$$\begin{bmatrix} \frac{4801}{12} & \frac{-9599}{24} & 0 & 0 & 0 & 1 & 0 \\ \frac{-9599}{24} & \frac{19253}{30} & \frac{-28987}{120} & 0 & 0 & 0 & 0 \\ 0 & \frac{-28987}{120} & \frac{7373}{30} & \frac{-95}{24} & 0 & 0 & 0 \\ 0 & 0 & \frac{-95}{24} & \frac{49}{6} & \frac{-95}{24} & 0 & 0 \\ 0 & 0 & 0 & \frac{-95}{24} & \frac{49}{12} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ \bar{u}_L \\ \bar{u}_R \end{bmatrix}$$

The resultant matrix is ill-conditioned, since the condition number is $4.696769356745330e + 03$.

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