Theoretica Homework

Coupled Problems
Assignment -1

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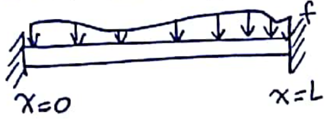
1 _ TRANSMISSION CONDITION:



The principal virtual Work states that the solution $v(x)$ satisfies:

$$EI\int_0^L \frac{d^2\delta u}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f \quad \begin{cases} \delta v(0) = \delta v(L) = 0 \\ \frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0 \end{cases}$$

BEAM Problem

A) SPACE OF FUNCTIONS:

First of All we consider that the L.h.s and the R.h.s of the equation must be bonded. So, the following condition must be fulfilled:

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} < \infty \quad \text{and} \quad \int_0^L \delta v f < \infty$$

So, the regularity condition for $L^2(\Omega)$ must be satisfied.

$$L^2(\Omega) = \left\{ p : \Omega \rightarrow \mathbb{R} \,/\, \int_\Omega p^2 = \|v\|^2_{L^2(\Omega)} < \infty \right\} \Rightarrow \begin{matrix} \delta v \in L^2(\Omega) & v \in L^2(\Omega) \\ \frac{d^2\delta v}{dx^2} \in L^2(\Omega) & \frac{d^2 v}{dx^2} \in L^2(\Omega) \end{matrix}$$

Let's see what happend. with First derivatives:

we know that $\underline{\nabla}\cdot(\delta v \underline{\nabla}\delta v) = \underline{\nabla}\delta v \cdot \underline{\nabla}\delta v + \delta v \underline{\nabla}\cdot(\underline{\nabla}\delta v) \Rightarrow$ Integrating $\Rightarrow$

$$\Rightarrow \int_\Omega \underline{\nabla}\cdot(\delta v \underline{\nabla}\delta v)d\Omega = \int_\Omega (\underline{\nabla}\delta v \cdot \nabla\delta v)d\Omega + \int_\Omega \delta v \underline{\nabla}\cdot(\underline{\nabla}\delta v)d\Omega \Rightarrow \text{In 1D.}$$

$$\Rightarrow \cancel{\delta v \frac{d\delta v}{dx}\Big|_\Gamma} = \int_\Omega \left(\frac{d\delta v}{dx}\right)^2 dx + \int_\Omega \delta v \frac{d^2\delta v}{dx^2} d\Omega \Rightarrow \int_\Omega \left(\frac{d\delta v}{dx}\right)^2 dx = -\int_\Omega \delta v \frac{d^2\delta v}{dx^2} d\Omega$$

0 $\delta v(L) = \delta(0) = 0$

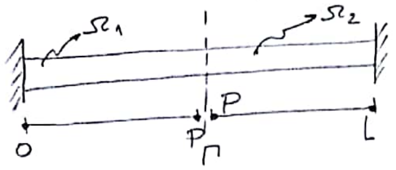
$\Rightarrow$ rewritting the equation above $\Rightarrow \int_\Omega |\underline{\nabla} v|^2 = -\underbrace{\int_\Omega \delta v \, \underline{\nabla}\cdot\nabla v \, d\Omega}_{\in L^2} \Rightarrow |\underline{\nabla} v|^2 \in L^2$

$\Rightarrow \frac{d\delta v}{dx} \in L^2$; in the same way $\frac{dv}{dx} \in L^2$.

AT THE END $v; \delta v \in H^2(\Omega)$; where $H^1(\Omega) = \left\{ p : \Omega \rightarrow \mathbb{R} \,/\, \int_\Omega p^2 < \infty ; \int_\Omega |\nabla p|^2 < \infty , \int_\Omega |\Delta v|^2 < \infty \right\}$

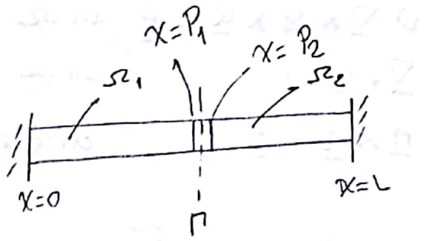
In particular, as $\delta v = 0$ on the Boundary, then $\delta v \in H_0^2$

B) If $[0,L]=[0,P]\cup(P,L]$, obtain the transmission condition at P implied by REGularity requeriments.



If our solution v is discontinouos across the point P (1D), then, v can not lie in $H^2(\Omega)$ and the gradient in the weak FORM could not be defined in $H^2(\Omega)$. Finally, the following condition on the interface must be satisfied, so-called Transmition CONDition:

$$[\![v]\!]_\Gamma = 0 \; ; \; [\![\nabla v]\!]_\Gamma = 0. \text{ in 1D.}$$



C) OBTain the transmission condition at P.

$$EI\frac{d^4v}{dx^4}=f \longrightarrow \delta v \longrightarrow \int_0^L EI\,\delta v\frac{d^4v}{dx^4}=\int_0^L \delta v f$$

1° Using Integration by part twice; then we obtain:

$$EI\left\{\delta v\frac{d^3v}{dx^3}\Big|_0^L-\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_0^L+\int_0^L\frac{d\delta^2 v}{dx^2}\frac{d^2v}{dx^2}\right.=\int_0^L\delta v f.$$

2° We split the whole Domain in two part. $x\in[0;P_1]$; $x\in(P_2;L]$; taking into account that $\delta v(0)=\delta v(L)=0$; $\frac{d\delta(0)}{dx}=\frac{d\delta(L)}{dx}=0$.

$$\Omega_1: \; EI\left\{\delta v\frac{d^3v}{dx^3}\Big]_{P_1}-\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big]_{P_1}+\int_0^{P_1}\frac{d\delta^2 v}{dx^2}\frac{d^2v}{dx^2}\right\}=\int_0^{P_1}\delta v f$$

$$\Omega_2: \; EI\left\{-\delta v\frac{d^3v}{dx^3}\Big]_{P_2}+\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big]_{P_2}+\int_{P_2}^{L}\frac{d\delta^2 v}{dx^2}\frac{d^2v}{dx^2}\right\}=\int_{P_2}^{L}\delta v f$$

Now; as the integrals are additives and $\Omega=\Omega_1\cup\Omega_2$.

$$EI\left\{\delta v\frac{d^3v}{dx^3}\Big]_{P_1}-\delta v\frac{d^3v}{dx^3}\Big]_{P_2}\right\}+EI\left\{\frac{d\delta v}{dx}\frac{d^2v}{dx^3}\Big]_{P_2}-\frac{d\delta v}{dx}\frac{d^2v}{dx^3}\Big]_{P_1}\right\}+EI\int_0^{P_1}\frac{d\delta^2 v}{dx^2}\frac{d^2v}{dx^2}+EI\int_{P_2}^{L}\frac{d\delta^2 v}{dx^2}\frac{d^2v}{dx^2}=$$

$$=\int_0^{P_1}\delta v f+\int_{P_2}^{L}\delta v f$$

As $P_1=P_2 \Rightarrow EI\int_0^{P_1}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx}+\int_{P_2}^{L}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx}=EI\int_0^{L}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx}$

$$\int_0^{P_1}\delta v f+\int_{P_2}^{L}\delta v f=\int_0^{L}\delta v f.$$

Then; $EI\,\delta v\left[\left.\frac{d^3v}{dx^3}\right]_{P_1}-\left.\frac{d^3v}{dx^3}\right]_{P_2}\right]=0$ and $EI\frac{d\delta}{dx}\left[\left.\frac{d^2v}{dx^2}\right]_{P_2}-\left.\frac{d^2v}{dx^2}\right]_{P_1}\right]=0$

Finally, $\left.\frac{d^2v}{dx^2}\right]_{P_1}=\left.\frac{d^2v}{dx^2}\right]_{P_2}$ or Bending moment, $M=-EI\frac{d^2v}{dx^2}$ must be continuous

and $\left.\frac{d^3v}{dx^3}\right]_{P_1}=\left.\frac{d^3v}{dx^3}\right]_{P_2}$ or shear Force, $Q=-EI\frac{d^3v}{dx^3}$

2- THE MAXWELL PROBLEM consists in finding a vector fiel $\underline{u}:\Omega\rightarrow\mathbb{R}^3$ such that:

$$\nu\,\underline{\nabla}\times\underline{\nabla}\times\underline{u}=\underline{f}\quad \text{in } \Omega$$
$$\underline{\nabla}\cdot\underline{u}=0\quad \text{in } \Omega$$
$$\underline{n}\times\underline{u}=\underline{0}\quad \text{on } \partial\Omega$$

1° We will take the maxwell problem and we will multiply it by the test function $\underline{v}$, and integrate it in Ω.

$$\int_\Omega \underline{v}\cdot(\nu\,\underline{\nabla}\times\underline{\nabla}\times\underline{u})=\int_\Omega\underline{v}\cdot\underline{f}$$

Now, we will use the following IDENTity. $\underline{\nabla}\cdot(\underline{B}\times\underline{C})=\underline{C}\cdot\underline{\nabla}\times\underline{B}-\underline{B}\cdot\underline{\nabla}\times\underline{C}$

Where $\underline{B}=\underline{\nabla}\times\underline{u}$; and $\underline{C}=\underline{v}$ $\Rightarrow$ $\underline{C}\cdot(\underline{\nabla}\times\underline{B})=\underline{\nabla}\cdot(\underline{B}\times\underline{C})+\underline{B}\cdot\underline{\nabla}\times\underline{C}$

$$\Rightarrow \int_\Omega\underline{C}\cdot(\nu\underline{\nabla}\times\underline{B})=\underbrace{\int_\Omega\nu\underline{\nabla}\cdot(\underline{B}\times\underline{C})}_{\text{Using DiV Theorem.}}+\int_\Omega\underline{B}\cdot(\underline{\nabla}\times\underline{C})=\int_\Gamma\nu\underline{n}\cdot\underset{①}{(\underline{B}\times\underline{C})}+\int_\Omega\underline{B}\cdot(\underline{\nabla}\times\underline{C})\Rightarrow$$

Using the next iDentity $\underline{n}\times(\underline{B}\times\underline{C})=(\underline{B}\cdot\underline{n})\times\underline{C}$ in ① $\Rightarrow\int_\Gamma\nu[\underline{n}\times\underline{B}]\cdot\underline{C}=\int_\Gamma\nu\underline{n}\cdot[\underline{B}\times\underline{C}$

$\Rightarrow \int_\Gamma\nu[\underline{n}\times\underline{B}]\cdot\underline{C}+\int_\Omega\underline{B}\cdot(\underline{\nabla}\times\underline{C})$ and recalling that $\underline{B}=\underline{\nabla}\times\underline{u}$; $\underline{C}=\underline{v}$

Finally, the weak Form of MAXWELL EQ. is:

$$\int_\Gamma\nu\,[\underline{n}\times(\underline{\nabla}\times\underline{u})]\cdot\underline{v}+\int_\Omega(\underline{\nabla}\times\underline{u})\cdot(\underline{\nabla}\times\underline{v})=\int_\Omega\underline{v}\cdot\underline{f}\quad\forall\, v\in H_{curl}.$$

$$H_{curl}=\left\{\underline{p}:\Omega\rightarrow\mathbb{R}^3 \,/\, \underline{p}\in L^2(\Omega);\ \underline{\nabla}\times\underline{p}\in L^2(\Omega)\right\}$$

Now, the following condition must be fulfilled; the regularity condition:

$$\int_{\Omega} |\nabla \times \underline{p}|^2 < \infty .$$

Using the stoke problem $\int_{\Gamma} (\underline{\nabla} \times \underline{P}) \cdot \underline{n} d\underline{s} = \oint_{L} \underline{p} \cdot \underline{t} d\underline{L}$ or $\int_{\Gamma} (\underline{\nabla} \times \underline{P}) \cdot \underline{n} ds = \oint_{L} (\underline{p} \times \underline{n}) d\underline{L}$

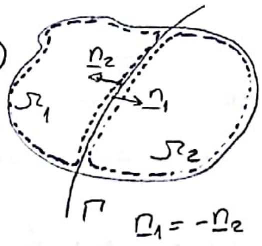
so, The transmition condition across the interfrece must be $[\![\underline{p} \times \underline{n}]\!] = \underline{0}$

or, in this problem $[\![\underline{u} \times \underline{n}]\!] = \underline{0}$.

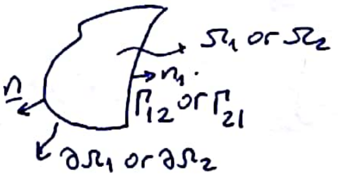
Next step, is split the Domain in two parts and apply the weak Form:

$$\int_{\Gamma_{12}} \nu [\underline{n}_1 \times (\underline{\nabla} \times \underline{u}_1)] \cdot \underline{v} + \int_{\partial\Omega_1} \nu [\underline{n}_1 \times (\underline{\nabla} \times \underline{u}_1)] \cdot \underline{v} + \int_{\Omega_1} (\underline{\nabla} \times \underline{u}) \cdot (\underline{\nabla} \times \underline{v}) = \int_{\Omega_1} \underline{v} \cdot f_1 \quad ①$$

$$\int_{\Gamma_{21}} \nu [\underline{n}_2 \times (\underline{\nabla} \times \underline{u}_2)] \cdot \underline{v} + \int_{\partial\Omega_2} \nu [\underline{n}_2 \times (\underline{\nabla} \times \underline{u}_2)] \cdot \underline{v} + \int_{\Omega_2} (\underline{\nabla} \times \underline{u}) \cdot (\underline{\nabla} \times \underline{v}) = \int_{\Omega_2} \underline{v} \cdot f_2 \quad ②$$



EACH SUBDOMAIN IS



Summing up ① and ② we recover the original weak form, but two terms remains. Using the fact that $\underline{n}_1 = -\underline{n}_2$

$$\int_{\Gamma_{21}} \nu [\underline{n} \times (\underline{\nabla} \times \underline{u}_2)] \cdot \underline{v} - \int_{\Gamma_{12}} \nu [\underline{n} \times (\underline{\nabla} \times \underline{u}_1)] \cdot \underline{v} = 0$$

The Equation above must satisfy the transmition condition across the interface

$$[\![\underline{p} \times \underline{n}]\!] = [\![\underline{n} \times \underline{p}]\!] = \underline{0} \quad \Rightarrow \text{if } \underline{p} = (\underline{\nabla} \times \underline{u}) \Rightarrow [\![\underline{n} \times (\underline{\nabla} \times \underline{u})]\!] = \underline{0}$$

Finally:

a) $\int_{\Gamma} \nu [\underline{n} \times (\underline{\nabla} \times \underline{u})] \cdot \underline{v} \, d\Gamma + \int_{\Omega} (\underline{\nabla} \times \underline{u}) \cdot (\underline{\nabla} \times \underline{v}) d\Omega$ weak form

$\forall \underline{v} \in H_{curl}$

b) $[\![\underline{n} \times \underline{u}]\!] = 0$ transmisstion condition across Γ.

c) $[\![\underline{n} \times (\underline{\nabla} \times \underline{u})]\!] = 0$ transmistion condition across Γ that follow by Imposing in the variational form of the problem.

3_ Navier-[redacted] equations for an elastic material can be written in three diffrent ways:

A) $-2\mu\,\underline{\nabla}\cdot(\underline{\underline{\varepsilon}}(\underline{u})) - \lambda\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b}$

$\underline{\underline{\varepsilon}}(\underline{u}) = \underline{\nabla}^S\underline{u} = \frac{1}{2}(\underline{\nabla}\underline{u} + \underline{\nabla}^T\underline{u})$

Symmetric part.

B) $-\mu\,\Delta\underline{u} - (\lambda+\mu)\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b}$

c) $\mu\,\underline{\nabla}\times(\underline{\nabla}\times\underline{u}) - (\lambda+2\mu)\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b}$

A) variational form of the previous equations: For simplicity we consider $\mu;\lambda$ cte.

$$\int_\Omega -2\mu\,\underline{v}\cdot\underset{①}{(\underline{\nabla}\cdot\underline{\nabla}^S\underline{u})} - \lambda\int_\Omega \underline{v}\cdot\underset{②}{\underline{\nabla}}(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{v}\cdot\rho\underline{b}$$

Using integration by part in ① and ② and Gauss' theorem.

① $\int_\Omega \underline{\nabla}\cdot(\underline{v}\cdot\underline{\nabla}^S\underline{u}) = \int_\Omega \underline{\nabla}\underline{v} : \underline{\nabla}^S\underline{u} + \int_\Omega \underline{v}\cdot\underline{\nabla}\cdot(\underline{\nabla}^S\underline{u})$

② $\int_\Omega \underline{\nabla}\cdot(\underline{v}(\underline{\nabla}\cdot\underline{u})) = \int_\Omega (\underline{\nabla}\cdot\underline{v})(\underline{\nabla}\cdot\underline{u}) + \int_\Omega \underline{v}\cdot\underline{\nabla}(\underline{\nabla}\cdot\underline{u})$

$$-2\mu\int_\Gamma \underline{v}\cdot(\underline{\nabla}^S\underline{u})\cdot\underline{n}\,d\Gamma + 2\mu\int_\Omega \underline{\nabla}\underline{v}:\underline{\nabla}^S\underline{u} - \lambda\int_\Gamma \underline{v}(\underline{\nabla}\cdot\underline{u})\cdot\underline{n}\,d\Gamma + \lambda\int_\Omega(\underline{\nabla}\cdot\underline{v})(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{v}\cdot\rho\underline{b}$$

$$\int_\Omega -\mu\,\underline{v}\cdot\underset{①}{\underline{\nabla}}\cdot(\underline{\nabla}\underline{u}) - \int_\Omega(\lambda+\mu)\,\underset{②}{\underline{v}}\cdot\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{v}\cdot\rho\underline{b}$$

Using integration by part and Gauss' theorem:

① $\int_\Omega \underline{\nabla}\cdot(\underline{v}\cdot\underline{\nabla}\underline{u}) = \int_\Omega \underline{\nabla}\underline{v} : \underline{\nabla}\underline{u} + \int_\Omega \underline{v}\cdot\underline{\nabla}^2\underline{u}$

② $\int_\Omega \underline{\nabla}\cdot(\underline{v}(\underline{\nabla}\cdot\underline{u})) = \int_\Omega(\underline{\nabla}\cdot\underline{v})(\underline{\nabla}\cdot\underline{u}) + \int \underline{v}\cdot(\underline{\nabla}^2\underline{u})$

$$-\mu\int_\Gamma \underline{v}\cdot(\underline{\nabla}\underline{u})\cdot\underline{n}\,d\Gamma + \mu\int_\Omega \underline{\nabla}\underline{v}:\underline{\nabla}\underline{u} - (\lambda+\mu)\int_\Gamma \underline{v}(\underline{\nabla}\cdot\underline{u})\cdot\underline{n}\,d\Gamma + (\lambda+\mu)\int_\Omega(\underline{\nabla}\cdot\underline{v})(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{v}\cdot\rho\underline{b}$$

Before starting with the third expression of the Navier's elastic equation, we are going to express $\mu(\underline{\nabla}\times\underline{\nabla}\times\underline{u})$ using the following identity.

$$\nabla^2\underline{u} = \underline{\nabla}(\underline{\nabla}\cdot\underline{u}) - \underline{\nabla}\times\underline{\nabla}\times\underline{u} \Rightarrow \underline{\nabla}\times\underline{\nabla}\times\underline{u} = \underline{\nabla}(\underline{\nabla}\cdot\underline{u}) - \nabla^2\underline{u}$$

But, It would be better if we prove the identity written above.

L.h.s $\longrightarrow$ $$\underline{\nabla}\times\underline{\nabla}\times\underline{u} = \hat{e}_i \times \frac{\partial}{\partial x_i}\left(\hat{e}^j \times \frac{\partial v_k}{\partial x_j}\hat{e}_k\right) = \hat{e}_i \times \frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}\varepsilon_{jkp}\hat{e}_p\right)$$

$$= \frac{\partial^2 v_k}{\partial x_i \partial x_j}\varepsilon_{jkp}\varepsilon_{ipl}\hat{e}_l$$

Now, we can express Levy-Civita tensor as $\varepsilon_{pjk}\,\varepsilon_{pli} = \delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}$.

$$\Rightarrow \left(\frac{\partial^2 v_k}{\partial x_i \partial x_j}\delta_{jl}\delta_{ki} - \frac{\partial^2 v_k}{\partial x_i \partial x_j}\delta_{ji}\delta_{kl}\right)\hat{e}_l = \left(\frac{\partial^2 v_k}{\partial x_k \partial x_l} - \frac{\partial^2 v_l}{\partial x_i \partial x_i}\right)\hat{e}_l$$

R.h.s

$$\nabla(\underline{\nabla}.u) = \hat{e}^i\frac{\partial}{\partial x_i}\left(\hat{e}^j.\frac{\partial v_k}{\partial x_j}\hat{e}_k\right) = \hat{e}^i\frac{\partial^2 v_k}{\partial x_i \partial x_j}\delta_{jk} = \frac{\partial^2 v_k}{\partial x_i \partial x_k}\hat{e}_i$$

$$\underline{\nabla}\cdot(\underline{\nabla}\,\underline{u}) = \hat{e}^i\cdot\frac{\partial}{\partial x_i}\left(\hat{e}_j\frac{\partial v_k}{\partial x_j}\hat{e}_k\right) = \frac{\partial^2 v_k}{\partial x_i \partial x_j}\delta_{ij}\hat{e}_k = \frac{\partial^2 v_k}{\partial x_i \partial x_i}\hat{e}_k$$

Putting L.h.s and R.h.s together, we get.

$$\left(\frac{\partial^2 v_k}{\partial x_k \partial x_l} - \frac{\partial^2 v_l}{\partial x_i \partial x_i}\right)\hat{e}_l = \frac{\partial^2 v_k}{\partial x_i \partial k}\hat{e}_i - \frac{\partial^2 v_k}{\partial x_i \partial x_i}\hat{e}_k$$ $\Rightarrow$ Finally we could prove that $\underline{\nabla}\times\underline{\nabla}\times\underline{u} = \underline{\nabla}(\underline{\nabla}.u) - \underline{\nabla}^2\underline{u}$

AFTER proving the identity written above, we cand Repleace the expression previously mentioned in equation "C":

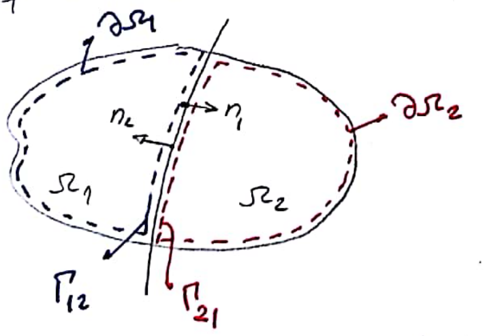
$$\mu\,\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) - \mu\nabla^2\underline{u} - (\lambda+2\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b} \Rightarrow$$

$$\Rightarrow -\mu\,\underline{\nabla}^2\underline{u} - (\lambda+\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b}$$ Same expression that "b"; then, same variation form.

The space functions will be $\forall v \in [H^1(\Omega)]$ and $\underline{u} \in [H^1(\Omega)]$.

3) Transmission condition across Γ.

Equal to other excersices, we will split the whole Domain in two parts.



We are going to use the eq. "B" in order to find the transmission condition on Γ.

in order to simplify operation; we will name the weak form of equation "B" as $a(\underline{v}, \underline{u})$

$$a(\underline{v}, \underline{u}) = -\mu \int_{\Gamma} \underline{v} \cdot (\nabla \underline{u}) \cdot \underline{n} \, d\Gamma + \mu \int_{\Omega} \nabla \underline{v} : \nabla \underline{u} - (\lambda + \mu) \int_{\Gamma} \underline{v} (\nabla \cdot \underline{u}) \cdot \underline{n} \, d\Gamma + (\lambda + \mu) \int_{\Omega} (\nabla \cdot \underline{v}) (\nabla \cdot \underline{u}) -$$

$$a(\underline{v}, u_1) - \mu \int_{\Gamma_{12}} \underline{v} \cdot (\nabla \underline{u}_1) \cdot \underline{n}_1 d\Gamma - (\lambda + \mu) \int_{\Gamma_{12}} \underline{v} (\nabla \cdot \underline{u}_1) \cdot \underline{n}_1 \, d\Gamma = \int_{\Omega_1} \underline{v} \cdot \rho b_1 \quad \text{in } \Omega_1$$

$$a(\underline{v}, u_2) - \mu \int_{\Gamma_{21}} \underline{v} \cdot (\nabla \underline{u}_2) \cdot \underline{n}_2 d\Gamma - (\lambda + \mu) \int_{\Gamma_{21}} \underline{v} (\nabla \cdot \underline{u}_2) \cdot \underline{n}_2 \, d\Gamma = \int_{\Omega_2} \underline{v} \cdot \rho b_2 \quad \text{in } \Omega_2$$

Summing the two subdomain up: and taking into account that $\underline{n}_1 = -\underline{n}_2$

$$a(\underline{v}, \underline{u}_1) + a(\underline{v}, u_2) + \mu \int_{\Gamma_{12}} \underline{v} \cdot (\nabla \underline{u}_1) \underline{n} \, d\Gamma - \mu \int_{\Gamma_{21}} \underline{v} \cdot (\nabla \underline{u}_2) \cdot \underline{n} \, d\Gamma + (\lambda + \mu) \int_{\Gamma_{12}} \underline{v} (\nabla \cdot \underline{u}_1) \cdot \underline{n} \, d\Gamma -$$

$$- \int_{\Gamma_{21}} \underline{v} (\nabla \cdot \underline{u}_2) \cdot \underline{n} \, d\Gamma = \int_{\Omega_1} \underline{v} \cdot \rho b_1 + \int_{\Omega_2} \underline{v} \cdot \rho b_2$$

We can see that we recover the original weak form. but with foor extra terms.

Since we used the fact that the integrals are additives, then

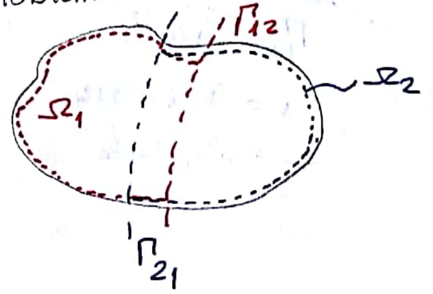
$$[\![(\nabla \underline{u}) \cdot \underline{n}]\!] = \underline{0} \quad \text{and} \quad [\![\nabla \cdot \underline{u}]\!] = 0$$

2. Domain Decomposition methods

1. Iteration-by-subdomain scheme

A) schwarz method

schwarz method implies over lapping, hence, each subdomain has a dirichet problem.



$\Omega_{12} = \bar{\Omega}_1 \cap \bar{\Omega}_2 \leftarrow$ over lapping region

Subdomain 1:

$$EI \frac{d^4 v_1^{(k)}}{dx^4} = f_1 \quad x \in [0, L_1]$$

$v_1^{(k)} = 0$ on $x = 0$

$\frac{d v_1^{(k)}}{dx} = 0$ on $x = 0$

$v_1^{(k)} = v_2^{(k-1)}$ on $x = L_1$

Subdomain 2:

$EI\, d^4 v_e^{(k)} = f_2 \quad x \in [L_2, L]$

$v_2^{(k)} = 0$ on $x = L$

$\frac{d v_1^{(k)}}{dx} = 0$ on $x = L$

$v_2^{(k)} = v_1^{(l)}$ on $x = L_2$

In order to use an additive Schwarz method, we have to set $l = k-1$. So, this two Domain can be computed in parallell.

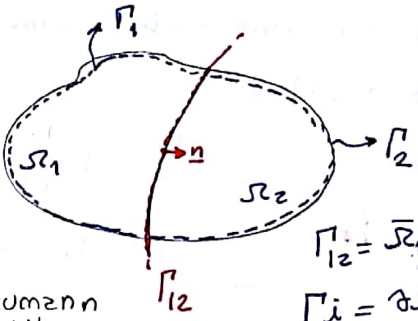
B) Algebraical version

$$\begin{bmatrix} A_{11} & A_{1\Gamma_{12}} \\ A_{\Gamma_{12}1} & A_{\Gamma_{12}\Gamma_{12}} \end{bmatrix} \begin{bmatrix} \underline{v}_1^{(k)} \\ v_{1\Gamma_{12}} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{\Gamma_{12}} \end{bmatrix}$$ for sub Domain Ω_1

$$\begin{bmatrix} A_{22} & A_{2\Gamma_{21}} \\ A_{\Gamma_{21}2} & A_{\Gamma_{21}\Gamma_{21}} \end{bmatrix} \begin{bmatrix} \underline{v}_2^{(k)} \\ v_{2\Gamma_{21}} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_{\Gamma_{21}} \end{bmatrix}$$ for subdomain Ω_2

where $u_{1\Gamma_{12}} = u_2^{k-1}$ on $x = L_1$ and $u_{2\Gamma_{21}} = u_1^{k-1}$ on $x = L_2$

2. Dirichlet–Neumann coupling: Maxwell Equations.



$\Gamma_{12} = \bar{\Omega}_1 \cap \bar{\Omega}_2$

$\Gamma_i = \partial\Omega_i \cap \partial\Omega$

$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$

A) **Sub Domain Ω_1**

$\nu \, \underline{\nabla} \times \underline{\nabla} \times u_1^{(k)} = f_1$ in Ω_1

$\underline{\nabla} \cdot u_1^{(k)} = 0$ in Ω_1

$\underline{n}_1 \times \underline{u}_1^{(k)} = 0$ on Γ_1

$\underline{n}_1 \times (\underline{\nabla} \times \underline{u}_1^{(k)}) = \underline{n}_1 \times (\underline{\nabla} \times u_2^{(k-1)})$ on Γ_{12} ← Neumann condition

Subdomain Ω_2

$\nu \, \underline{\nabla} \times \underline{\nabla} \times u_2^{(k)} = f_2$ in Ω_2

$\underline{\nabla} \cdot u_2^{(k)} = 0$ in Ω_2

$\underline{n}_2 \times u_2^{(k)} = 0$ on Γ_2

$\underline{n}_2 \times u_2^{(k)} = \underline{n}_2 \times \underline{u}_1^{(\ell)}$ on Γ_{12} ← Dirichlet condition

$\ell = k-1$ is JAcobian descomposition method ; $\ell = k$ is Gauss-Seidel decomp. method.

B) Steklov-Poincaré operator.

In order to obtain a Steklov-Poincaré operator (Direct Method) , we are going to propose the following expression for the solution $\underline{u}$.

$u_i = u_i^0 + \tilde{u}_i \quad i = 1, 2.$

SubDomain Ω_i

$$\left.\begin{aligned} &\nu \, \underline{\nabla} \times \underline{\nabla} \times u_i^0 = f_i && \text{in } \Omega_i \\ &\underline{\nabla} \cdot \underline{u}_i^0 = 0 && \text{in } \Omega_i \\ &\underline{n}_i \times u_i^0 = \underline{0} && \text{on } \Gamma_i \\ &\underline{n}_i \times u_i^0 = \underline{0} && \text{on } \Gamma_{12} \end{aligned}\right\} \qquad \begin{aligned} &\nu \, \underline{\nabla} \times \underline{\nabla} \times \tilde{u}_i = 0 && \text{in } \Omega_i \\ &\underline{\nabla} \cdot \tilde{\underline{u}}_i = 0 && \text{in } \Omega_i \\ &\underline{n}_i \times \tilde{u}_i = 0 && \text{on } \Gamma_i \\ &\underline{n}_i \times \tilde{u}_i = \varphi && \text{on } \Gamma_{12} \end{aligned}$$

" φ (unknown) must be satisfis the 2° transmission condition "

$$\underline{n} \times (\underline{\nabla} \times \underline{u}_1) = \underline{n} \times (\underline{\nabla} \times \underline{u}_2) \Rightarrow \underline{n} \times (\underline{\nabla} \times (\underline{u}_1^0 + \tilde{u}_1)) = \underline{n} \times (\underline{\nabla} \times (u_2^0 + \tilde{u}_2))$$

$$\Rightarrow \underbrace{\underline{n} \times (\underline{\nabla} \times \tilde{u}_1) - \underline{n} \times (\underline{\nabla} \times \tilde{u}_2)}_{\mathcal{S}} = \underbrace{\underline{n} \times (\underline{\nabla} \times u_2^0) - \underline{n} \, (\underline{\nabla} \times u_1^0)}_{\text{Known} (\mathcal{C})}$$

$\Rightarrow$ Finally $\boxed{\mathcal{S} \varphi = \mathcal{C}}$ ← Direct Method.

c) Algebraical version.

the matrix version of the whole domain will be:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} & 0 \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)}+A_{\Gamma\Gamma}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & A_{\Gamma I}^{(2)} & A_{II}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ u_{\Gamma_{12}} \\ u_I^{(2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{\Gamma_{12}} \\ f_2 \end{bmatrix}$$

For the first subdomain we have a Neumann problem and It can be written as:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{(1)k} \\ u_{\Gamma_{12}}^{(k)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_{\Gamma_{12}} - A_{\Gamma I}^{(2)} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_{\Gamma_{12}}^{k-1} \end{bmatrix}$$

For the second subdomain we have a Dirichlet problem and It can be written as:

$$A_{II}^{(2)} u_2^{(k)} = f_2 - A_{I\Gamma}^{(2)} u_\Gamma^{k-1}$$

3). consider the problem of finding $u : \Omega \rightarrow \mathbb{R}$ /

$$-k \nabla^2 \underline{u} = f \text{ in } \Omega \qquad k>0$$

$$\underline{u} = 0 \text{ on } \partial\Omega$$

A) Dirichlet-Robin.

Subdomain Ω_1: Dirichlet.

$$-k \nabla^2 \underline{u}_1^{(k)} = f_1 \text{ in } \Omega_1$$

$$\underline{u}_1 = 0 \text{ on } \Gamma_1$$

$$\underline{u}_1^{(k)} = u_2^{(\ell)} \text{ on } \Gamma_{12}$$

subdomain Ω_2: Robin.

$$-k \nabla^2 \underline{u}_2^{(k)} = f_2 \text{ in } \Omega_2$$

$$\underline{u}_2 = 0 \text{ in } \Omega_2$$

$$k \frac{\partial \underline{u}_2^{(k)}}{\partial n} + \gamma_2 \underline{u}_2^{(k)} = k \frac{\partial u_1^{(\ell)}}{\partial n} + \gamma_1 u_1^{(\ell)} \text{ on } \Gamma_{12}$$

if $\ell = k-1$ Jacobi scheme, elseif $\ell = k$ Gauss seidel scheme.

Condition: these two equation must be linear independence; so, $\gamma_1 \neq \gamma_2$ and $\gamma_1 + \gamma_2 > 0$.

B) Matrix version of the previous scheme

Subdomain Ω_1: Dirichlet;

After integration by parts and considering w_1 as test function ($w_1 = 0$ on Dirichlet boundary)

$$\int_{\Omega_1} k \nabla w_1 \nabla u_1 = \int_{\Omega_1} w_1 f_1 \quad \text{in } \Omega_1$$

Subdomain Ω_2: Robin.

$$\int_{\Omega_2} k \nabla w_2 \nabla u_2 - \int_{\Gamma_{12}} k w_2 \nabla u_2 \cdot \underline{n} = \int_{\Omega_2} w_2 f_2 \text{ in } \Omega_2$$

$$\int_{\Omega_2} k \nabla w_2 \nabla u_2 - \int_{\Gamma_{12}} k w_2 \frac{\partial u_2}{\partial n} = \int_{\Omega_1} w_2 f_2$$ → using the flux expression found for Robin

$$\Rightarrow \int_{\Omega_2} k \nabla w_2 \nabla u_2 + \int_{\Gamma_{12}} k \gamma_2 w_2 u_2 = \int_{\Omega_2} w_2 f_2 + \int_{\Gamma_{12}} k\gamma_1 w_2 \frac{\partial u_1}{\partial n} + \int_{\Gamma_{12}} k \gamma_2 w_2 u_1$$

Now, we are going to name :

$$A = \int_{\Omega} k \nabla w \nabla u \quad \text{and} \quad M = \int_{\Omega} w \cdot u$$

Finally, the matrix expression for subdomain Ω_1 and Ω_2 are:

$$A_{II}^{(1)} u_I^{(1)k} = F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(\ell)} \qquad \text{in } \Omega_1$$

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} + \gamma_2 M_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} + \gamma_2 M_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(2)k} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} F_2 \\ f_\Gamma - \left[A_{\Gamma I}^{(1)} - \gamma_1 M_{\Gamma I}^{(1)}\right] u_I^{(1)k-1} - \left(A_{\Gamma\Gamma}^{(1)} - \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) u_\Gamma^{k-1} \end{bmatrix} \text{ in } \Omega_2$$

c) schur complement.

for Ω_1 ⟶ $u_I^{(1)k} = A_{II}^{(1)-1}\left(F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(\ell)}\right)$ ①

for Ω_2 ⟶ $u_I^{(2)k} = A_{II}^{(2)-1}\left(F_2 - A_{I\Gamma}^{(2)} u_\Gamma^{(k)}\right)$ ②, Matrix-vector product of first row.

Doing Matrix-vector product of the second row (for Ω_2) and using ②, we get:

$$\left[-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(2)}\right] u_\Gamma^{(k)} = F_\Gamma - A_{\Gamma I}^{(1)} u_I^{(1)k-1} - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{k-1} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$$

Using ① in the equation above, we get.

$$\left[-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(2)}\right] u_\Gamma^{(k)} = F_\Gamma - A_{\Gamma I}^{(1)} A_{II}^{(1)-1}\left(F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(\ell)}\right) - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{k-1} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$$

AFTER SOME ALGEBRA AND schrink related with $u_\Gamma^{(k)}$; the we get:

$$\underbrace{\left(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)}\right)}_{S} \underbrace{u_\Gamma^{k}}_{U_\Gamma} = \underbrace{F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1}_{G}$$

↑ Schur complement

$$\boxed{S\, U_\Gamma = G}$$

the script version of Steklov-Poincaré, is the Schur complement

D) Preconditioner foz the schur complement.

In order to obtain the preconditioner, we have to write the system using the following expression:

$$u_\Gamma^k = u_\Gamma^{k-1} + S_*^{-1}(F - Su_\Gamma^{k-1}).$$ Where S_* is the preconditioner.

Using the expression written in c, we can compute the following:

$$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)} \text{ and } S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} \text{ such that } S_1 + S_2 = S.$$

$$G = F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1$$

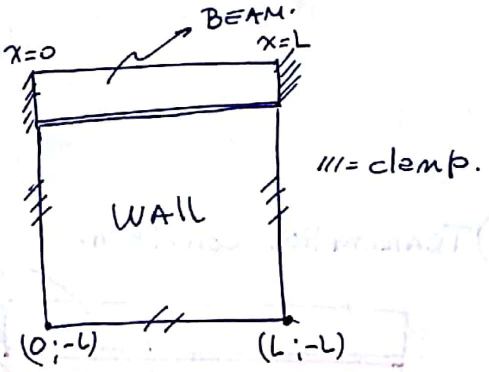
$$Su_\Gamma = G$$

$$\Rightarrow (S_1 + S_2) u_\Gamma = G \Rightarrow S_2 u_\Gamma = G - S_1 u_\Gamma \quad \text{but } S_1 = S - S_2 \Rightarrow S_2 u_\Gamma^k = G - Su_\Gamma^{k-1} + S_2 u_\Gamma^{k-1}$$

$$\Rightarrow u_\Gamma^k = \underbrace{S_2^{-1} S_2}_{I} u_\Gamma^{(k-1)} + S_2^{-1}(G - Su_\Gamma^{k-1}) \Rightarrow u_\Gamma^{(k)} = u_\Gamma^{(k-1)} + S_2^{-1}(G - Su_\Gamma^{k-1})$$

Finally S_2 is the preconditioner

3. Coupling of heterogeneous problems



A) USING HOOK'S LAW, we get:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

In terms of desplacement we get:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{bmatrix}$$

ow, we use THE MOMENTUM EQUETION

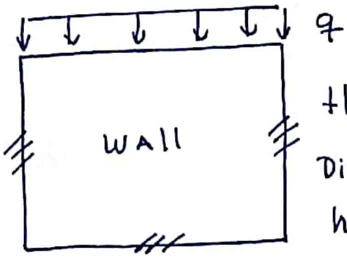
$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b} = 0$$

.FTER SOME AlGEBRA, WE obtein

$$\frac{\partial^2 u}{\partial x^2} + \frac{\upsilon \partial^2 v}{\partial x \partial y} + \frac{1-\upsilon}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right) + b_x = 0$$

$$\frac{\upsilon \partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{1-\upsilon}{2}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) + b_y = 0$$

B)



thickness = h

Distributed load along the top edge is

$h \times \sigma_y \big|_{y=0}$

1° the governing equation is: $EI\frac{d^4v}{dx^4} = f$

Now, Due to the wall, it would exist a distributed load along the top edge. so, we have to add this Distribute load into the governing Equation.

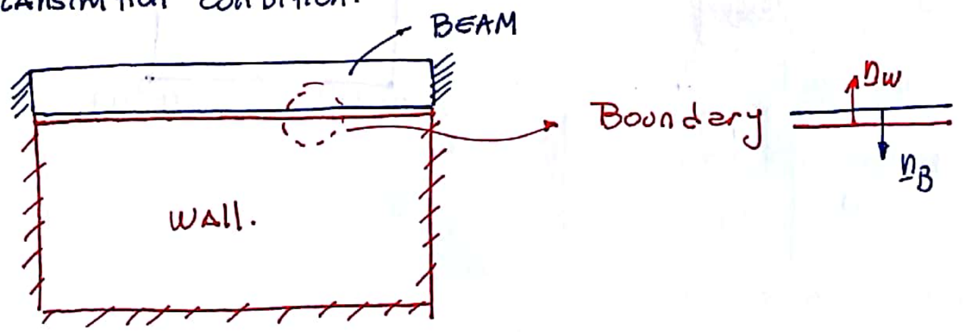
$$EI\frac{d^4v}{dx^4} = f - h \times \sigma_y \Big|_{y=0}$$

Taking into account the Equation written in A, we get:

$$\sigma_y = \frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Big|_{y=0}$$

Finally $$EI\frac{d^4v}{dx^4} + h \times \left[\frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\Big|_{y=0} = f$$

c) Transmition condition.



$n_B = -n_w$

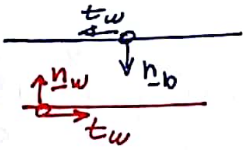
THE FIRST TRANSMITION CONDITION will be: VERTICAL DESPLACEMENT MUST BE THE SAME:

$$[\![v]\!] = 0$$

THE SECOND TRANSMITION CONDITION will be: NORMAL TRACCTION ONTOP OF THE WALL must be the same:

$$[\![\underline{n} \cdot \nabla \cdot \underline{\sigma}]\!]\Big|_{y=0} = 0$$

D) First of all, we Define an intrisec coordinat System on the Boundary.



t: tangent axis
n: normal axis

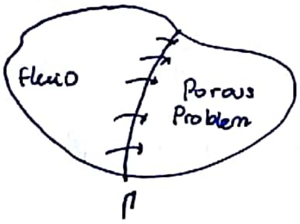
Since we consider free-slip condition in the contact surface. then the transmission condition of the horizontal displacement will be:

$[\![u]\!] \neq 0$ there will be Different horizontal Displacements. And the Transmission condition for the horizontal traction will be:

$[\![\underline{t} \cdot \underline{\nabla} \cdot \sigma]\!] \neq 0$ There will be Different horizontal traction.

2) Stokes – Darcy coupled problem:

A)



STOKES (S) Domain:

$$\begin{cases} -\mu \Delta \underline{u}_s + \nabla p_s = f \\ \underline{\nabla} \cdot \underline{u}_s = 0 \end{cases}$$

Darcy (D) Domain:

$$\begin{cases} \underline{\underline{K}}^{-1} \cdot \underline{u}_D + \underline{\nabla} \varphi = 0 \\ \underline{\nabla} \cdot \underline{u}_d = 0 \end{cases}$$

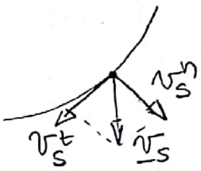
Weak form of Stokes Problem:

$$\int_{\Omega_s} \mu \, \underline{\nabla} \underline{v}_s : \underline{\nabla} \underline{u}_s \, d\Omega - \int_{\Omega_s} p_s \underline{\nabla} \cdot \underline{v}_s \, d\Omega - \int_{\partial\Omega_s} \underline{v}_s \cdot [\underline{n} \cdot (-p_s \underset{①}{\underline{\underline{I}}} + \nu \underline{\nabla}^s \underline{u}_s)] \, d\Omega = \int_{\Omega_s} \underline{v}_s \cdot \underline{f} \, d\Omega$$

WEAK Form of Darcy Problem:

$$\int_{\Omega_D} \underline{v}_D \cdot \underline{\underline{K}}^{-1} \cdot \underline{u}_D - \int_{\Omega_D} \varphi \, \underline{\nabla} \cdot \underline{v}_D + \int_{\partial\Omega_D} \underline{v}_D \cdot \underset{②}{\underline{n}} \varphi . = 0$$

Now, let's split $\underline{v}_s$ and $\underline{v}_D$.



$$\left. \begin{array}{l} \underline{v}_s = v_s^n \cdot \underline{n} + v_s^t \, \underline{t} \\ \underline{v}_D = v_D^n \, \underline{n} + v_D^t \, \underline{t} \end{array} \right\} \text{AND } v_s^n = v_D^n$$

If we take term ① $\Rightarrow \underline{v}_s \cdot [\underline{n} \cdot (-p_s \underline{\underline{I}} + \mu \underline{\nabla}^s \underline{u}_s)] = v_s^n p_s - \mu \underline{n} \cdot \underline{\nabla}^s \underline{u}_s \cdot \underline{n} \, v_s^n -$
$- \mu \underline{n} \cdot \underline{\nabla}^s \underline{u}_s \cdot t \, v_s^t$ ③

If we take term ② $\Rightarrow \underline{v}_D \cdot \underline{n} \cdot \varphi \underline{\underline{I}} = v_D^n \varphi$

Since $v_S^n = v_D^n \Rightarrow v_s^n [p_s - \mu \underline{n} \cdot \underline{\nabla}^s \underline{u}_s \cdot n] = v_D^n \varphi \Rightarrow \boxed{\varphi = p_s - \mu \underline{n} \cdot \underline{\nabla}^s \underline{u}_s \cdot n}$

AND For the tangent component of ③

$$\mu (\underline{n} \cdot \underline{\nabla}^s \underline{u}_s) \cdot \underline{t} = - \frac{\alpha_{BS}}{\sqrt{K}} (\underline{u}_s - \underline{u}_D) \cdot \underline{t} \qquad \text{with } K = \underline{t} \cdot \underline{\underline{K}} \cdot \underline{t}$$

Finally : The weak form of the problem will be:

$$\int_{\Omega_s} \mu \underline{\nabla} \underline{u}_s : \underline{\nabla} v \, d\Omega + \int_{\Gamma} v_s^t \frac{\alpha_{ps}}{\sqrt{K}} (\underline{u}_s - \underline{u}_D) \cdot \underline{t} \, d\Gamma - \int_{\Omega} p_s \underline{\nabla} \cdot \underline{v} \, d\Omega - \int_{\Gamma} v_s^N \underline{n} \cdot [-p_s \underline{\underline{I}} + \mu \underline{\nabla} \underline{u}_s] d\Gamma = \int_{\Omega} \underline{v} \cdot \underline{f} \, d\Omega$$

$$-\int_{\Omega} q_s \underline{\nabla} \cdot \underline{u}_s = 0$$

The Algebraical formulation is:

$A^S_{II} u_I + A^S_{I\Gamma} u_\Gamma + B^{TS} P_{SI} = F^S_I$ For the interior Ω_s

$A^S_{\Gamma I} u_I + A^S_{\Gamma\Gamma} u_\Gamma + B^S_{I\Gamma} P_S + M^d_{\Gamma\Gamma} P_{d\Gamma} = F^S_\Gamma$ For the boundary $\partial\Omega_s$

$A^D_{II} P_{DI} + A^D_{I\Gamma} P_{D\Gamma} = F^D_I$ For the Boundary $\partial\Omega_D$

$A^{TD}_{I\Gamma} P_{DI} + A^D_{\Gamma\Gamma} P_{D\Gamma} - M^{TD}_{\Gamma\Gamma} u_\Gamma = F^D_\Gamma$ For interior nodes on Γ

$B^S_{II} u_I + B^S_{I\Gamma} u_\Gamma = f_I$

In Algebraical Form:

$$\begin{bmatrix} A^S_{II} & B^{TS} & A^S_{I\Gamma} & 0 & 0 & 0 \\ B^S_{II} & 0 & B^S_{I\Gamma} & 0 & 0 & 0 \\ A^S_{\Gamma I} & B^{TS}_{I\Gamma} & A^S_{\Gamma\Gamma} & M^D_{\Gamma\Gamma} & 0 & 0 \\ 0 & 0 & 0 & -M^{TD}_{\Gamma\Gamma} & A^D_{\Gamma\Gamma} & A^{TD}_{I\Gamma} \\ 0 & 0 & 0 & 0 & A^D_{I\Gamma} & A^D_{II} \end{bmatrix} \begin{bmatrix} u_I \\ P_{SI} \\ u_\Gamma \\ P_{D\Gamma} \\ P_{DI} \end{bmatrix} = \begin{bmatrix} F^S_\Gamma \\ F_I \\ F^S_\Gamma \\ F^{D'}_\Gamma \\ F^D_I \end{bmatrix}$$

B)

DARCY PROBLEM (NEUMANN)

$$\begin{bmatrix} A^D_{\Gamma\Gamma} & A^{TD}_{I\Gamma} \\ A^D_{I\Gamma} & A^D_{II} \end{bmatrix} \begin{bmatrix} P^D_\Gamma \\ P^D_I \end{bmatrix}^{k+1} = \begin{bmatrix} F^D_\Gamma + M^{TD}_{\Gamma\Gamma} u^k_\Gamma \\ F^D_I \end{bmatrix}$$

STOKES PROBLEM (DIRICHLET)

$$\begin{bmatrix} A^S_{II} & B^{TS} & A^S_{I\Gamma} \\ B^S_{II} & 0 & B^S_{I\Gamma} \\ A^S_{I\Gamma} & B^{TS}_{I\Gamma} & A^S_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ P_{SI} \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} F^S_I \\ f_I \\ F^S_\Gamma - M^D_{\Gamma\Gamma} P^\ell_{D\Gamma} \end{bmatrix} \qquad \ell = \begin{cases} k+1 & \text{GAUSS - SEIDEL} \\ k & \text{JACOBI} \end{cases}$$

4. MONOLITHIC AND PARTITIONED SCHEMES IN TIME

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = f & \text{in } [0,1] \\ u(x=0, t) = 0 & \text{on } \Gamma_D \\ u(x=1, t) = 0 & \text{on } \Gamma_D \\ u(x, t=0) = 0 & \text{on } \Omega \text{ at } t \end{cases}$$

1) USING BACK DIFFERENCES IN TIME (BDF1), we will obtain the Discret Form of the Equation.

1° using θ-family methods. with $\theta = 1$

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u^n_t \qquad \text{with } \Delta u = u^{n+1} - u^n$$

$$\Delta u_t = u^{n+1}_t - u^n_t$$

Now, we are going to use the governing equation. $u_t = f + k\nabla^2 u$

At the end. $$\frac{\Delta u}{\Delta t} - \theta k \Delta(u_{xx}) = \theta f^{n+1} + 1 - \theta f^n + k u^n_{xx}$$

ONCE obtained the time Discretization, we will use space Discretization.

$$\int_\Omega w \frac{\Delta u}{\Delta t} d\Omega + \theta \int_\Omega \frac{dw}{dx}\frac{d(\Delta u)}{dx} = \theta \int_\Omega w f^{n+1} + (1-\theta)\int_\Omega w f^n - \int_\Omega \frac{dw}{dx}\frac{d(u^n)}{dx}$$

Taking into account $\theta = 1$; THE MATRIX FORM IS:

$$(\tilde{M} + \Delta t \tilde{K})\Delta \bar{u} = \Delta t \bar{f}^{n+1} - \Delta t \tilde{K}\bar{u}^n \qquad \Delta \bar{u} = \bar{u}^{n+1} - \bar{u}^n$$

$$\tilde{M} = \sum_e \int_{\Omega e} N_i N_j \; ; \; \tilde{K} = \sum_e \int_{\Omega e} \frac{dN_i}{dx}\frac{dN_j}{dx} \; ; \; f = \sum_e \int_{\Omega e} N_i f$$

2)

- SUBDOMAIN Ω_1

$$\left(v_1, \frac{\partial u_1}{\partial t}\right) + \left(\frac{\partial v_1}{\partial x}, \frac{\partial u_1}{\partial x}\right) - \left(v_1, \frac{\partial u_1}{\partial x}\right)_{\Gamma_{interface}} = (v_1, f_1)$$

- ON the other HAND, WE HAVE THE EQUATION FOR SUBDOMAIN 2. HERE WE USE THE FACT THAT:

$$k\frac{\partial u_2}{\partial x} = -k\frac{\partial u_1}{\partial x}$$

$$\left(v_2, \frac{\partial u_2}{\partial x}\right) + \left(\frac{\partial v_2}{\partial x}, \frac{\partial u_2}{\partial x}\right) - \left(v_2, \frac{\partial u_2}{\partial x}\right)_{\Gamma_{interface}} = (v_2, f_2)$$

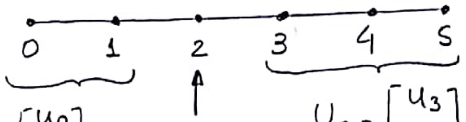
- Since the mesh match at the interface and we are using the same interpolation space v for u_1 and u_2; and ($v = 0$ on Γ_D)

$$\left(v, k\frac{\partial u_2}{\partial x}\right)_{\Gamma_{interf}} = -\left(v, k\frac{\partial u_1}{\partial x}\right)\Big|_{\Gamma_{interf.}} \Rightarrow \left(v, k\frac{\partial u_1}{\partial x}\right)\Big|_{\Gamma_{interf}} = \left(v, \frac{\partial u_1}{\partial t}\right) + \left(\frac{\partial v}{\partial x}, \frac{\partial u_1}{\partial x}\right) - (v, f_1)$$

$$\left(v, \frac{\partial u_2}{\partial t}\right) + \left(\frac{\partial v}{\partial x}, \frac{\partial u_2}{\partial x}\right) + \left(v, \frac{\partial u_1}{\partial t}\right) + \left(\frac{\partial v}{\partial x}, \frac{\partial u_1}{\partial x}\right) = (v, f_1) + (v, f_2)$$

There is no boundary integrals are required at the interface.

3) Dirichlet to Neumann



$$U_I = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \qquad U_\Gamma = u_2 \qquad U_2 = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

If we use the matrix expression written in 1). then, we obtain:

$$\tilde{A} = \frac{M}{\Delta t} + K \qquad \tilde{f} = \left(f + \frac{M}{\Delta t}\right) U^n \Rightarrow A \Delta U = \tilde{f}$$

FOR SUBDOMAIN Ω_1.

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \Delta U_I^{(1)} \\ \Delta U_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{f}_I^{(1)} \\ \tilde{f}_\Gamma^{(1)} \end{bmatrix} \qquad \begin{aligned} \Delta U_I^{(1)} &= U_I^{n+1} - U_I^n \quad ① \\ \Delta U_\Gamma &= U_\Gamma^{n+1} - U_\Gamma^n = u_2^{n+1} - u_2^n \quad ② \end{aligned}$$

$\Delta U_I^{(1)} = A_{II}^{(1)-1} \tilde{f}_I^{(1)} - A_{II}^{(1)-1} A_{I\Gamma}^{(1)} \Delta U_\Gamma$ using ① and ②, then we obtain:

$$\boxed{U_I^{n+1\,(1)} = A_{II}^{(1)-1} \tilde{f}_I^{(1)} - A_{II}^{(1)-1} A_{I\Gamma}^{(1)} U_\Gamma^{n+1} + A_{II}^{(1)-1} A_{I\Gamma}^{(1)} U_\Gamma^n + U_I^{n\,(1)}}$$

4) Neumann–Dirichlet for the right subDomain.

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta U_I^{(2)} \\ \Delta U_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{f}_I^{(2)} \\ \tilde{f}_\Gamma^{(2)} + t \end{bmatrix}$$

where $t = \int_\Gamma w k \frac{\partial U_\Gamma^{n+1}}{\partial x}$

we can express. $U_\Gamma^{n+1} = \tilde{f}_\Gamma^{(1)} - A_{\Gamma\Gamma}^{(1)} U_\Gamma^n - A_{\Gamma I}^{(1)} U_I^{(1)\,n+1}$

ONCE subDomain Ω^1 is solved, then we can solve subdomain Ω_2

5) ITERATIVE Algorithm.

SubDomain Ω_1 (LEFT)

$$\begin{cases} \partial_t u_1^{(k+1)} - k \dfrac{\partial^2 u_1^{(k+1)}}{\partial x^2} = f^{(1)} & \text{in } \Omega_1 \\ u_1^{k+1} = u_2^k & \text{on } \Gamma_{\text{interface}}. \\ u_1(x=0) = 0 & \Gamma_D \end{cases}$$

SubDomain Ω_2 (Right)

$$\begin{cases} \partial_t u_2^{(k+1)} - k \dfrac{\partial^2 u_2^{(k+1)}}{\partial x^2} = f^{(2)} & \text{in } \Omega_2 \\ k \dfrac{\partial u_2^{(k+1)}}{\partial x} = -k \dfrac{\partial u_1^k}{\partial x} & \text{on } \Gamma_{\text{interface}} \\ u_2(x=1) = 0 & \Gamma_D \end{cases}$$

ALGEBRAIC VERSION

$$U_{I(1)}^{(n+1)(k+1)} = A_{II}^{-1(1)} \tilde{f}_I^{(1)} - A_{II}^{-1(1)} A_{I\Gamma}^{(1)} U_\Gamma^{(n+1)(k)} + A_{II}^{-1(1)} A_{I\Gamma}^{(2)} U_\Gamma^n + U_I^{n\,(1)} \quad (\Omega_1)$$

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} U_{I(2)}^{(n+1)(k+1)} \\ U_\Gamma^{(n+1)(k+1)} \end{bmatrix} = \begin{bmatrix} \tilde{f}_I^{(2)} \\ \tilde{f}_\Gamma^{(2)} + t \end{bmatrix} - \begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} U_{I(2)}^{n,k} \\ U_{\Gamma(2)}^{n,k} \end{bmatrix} \quad (\Omega_2)$$

$$t = \tilde{f}_\Gamma^{(1)} - A_{\Gamma I}^{(1)} U_{I(1)}^{n+1,k} - A_{\Gamma\Gamma}^{(1)} U_\Gamma^{n,k}$$

6) The substitution scheme:

Here; each subdomain is sdved separatelly:

Subdomain Ω_1 (left)

$$\begin{cases} \partial_t u^{(i+1)} - k \dfrac{\partial^2 u_1^{(k+1)}}{\partial x^2} = f^2 \\ u_1^{(i+1)} = u_2^{(i)} \quad \text{on } \Gamma_{interface} \\ u_1^{(1)}(x=0) = 0 \quad \text{on } \Gamma_D \end{cases}$$

Subdomain Ω_2 (right)

$$\begin{cases} \partial_t u_2^{(i+1)} - k \dfrac{\partial^2 u_2^{(i+1)}}{\partial x^2} = f^2 \\ k \dfrac{\partial u_2^{(i+1)}}{\partial x} = - k \dfrac{\partial u_1^{(i+1)}}{\partial x} \quad \text{on } \Gamma_{interface} \\ u_2(x=1) = 0 \end{cases}$$

The Algebraical expressions are equal to the 5° point, the only Difference is $t = \tilde{f}_\Gamma^{(1)} - A_{\Gamma I}^{(1)} U_I^{(n+1),(k+1)} - A_{\Gamma\Gamma}^{(1)} U_\Gamma^n$

7) Nitche's method in Veriational form:

$$(v_1, \frac{\partial u_1}{\partial x}) + (\nabla v_1 \nabla u) + \alpha (v_1, u_1)_\Gamma - (v_1 \underline{n} \cdot k \underline{\nabla} u_1)_\Gamma - (u_1, \underline{n} \cdot k \nabla u_1)_\Gamma = (v_1, f) + \alpha (v_1, u_D)_\Gamma - $$
$$- (u_D, \underline{n} \cdot k \underline{\nabla} v_1)_\Gamma$$

The matrix form of the verizational form written above is:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} + \alpha M - B - BT \end{bmatrix} \begin{bmatrix} \Delta U_I^{(1)} \\ \Delta U_\Gamma^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{f}_I^{(1)} \\ \tilde{f}_\Gamma^{(1)} + \alpha M U_D - B^T U_D \end{bmatrix}$$

$$M = \int_\Gamma N^T N \; ; \quad B = \int_\Gamma N^T \frac{dN}{dx} \; ; \quad \Delta U = U^{n+1} - U^n$$

U_D: Dirichlet Data from the right subdomain

Pro:

1- MATRIX ON l.h.S is SPD.

2- In general, α does not require a very large number.

Cons:

1- If α increases, the condition number increases.

5. OPERATOR splitting techniques

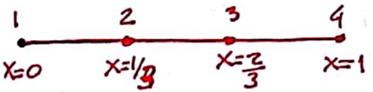
$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f \quad in\ [0,1]$$

$$\begin{cases} u(x=0, t)=0 \\ u(x=1, t)=0 \\ u(x, t=0)=0 \end{cases}$$

$k=1$ 3 element will be used

$a_x=1$

$f=1$



$\ell^{(e)} = h = \frac{1}{3}$

- Remark: It will be necessery to use method in order to solve the problem and plot splitting error vs time step Δt.

1.) We are consider BDF ; $\theta = 1$.

$$\frac{\Delta u}{\Delta t} + (\underline{a} \cdot \underline{\nabla} - \nabla^2)\Delta u = f^{n+1} - (\underline{a} \cdot \underline{\nabla} - \underline{\nabla}^2)u^n$$

AFTER SPATIAL DISCRETIZATION WE OBTAIN:

$$\left(\frac{\tilde{M}}{\Delta t} + \tilde{C} + \tilde{K}\right) u^{n+1} = f^{n+1} + \frac{M}{\Delta t} u^n$$

linear shape functions:

$$N_1 = \frac{1}{2}(1-\xi) \ ; \ N_2 = \frac{1}{2}(1+\xi)$$

$$f^e = \frac{\ell^{(e)}}{2}\int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} d\xi = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

$$M^e = \int_{-1}^{1} \frac{\ell^e}{2} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \left[\frac{1}{2}(1-\xi) \ \ \frac{1}{2}(1+\xi) \right] d\xi = \begin{bmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix}$$

$$K^e = \int_{-1}^{1} \frac{\ell^e}{2} \begin{bmatrix} \frac{-1}{\ell^e} \\ \frac{1}{\ell^e} \end{bmatrix} \left[\frac{-1}{\ell^e} \ \ \frac{1}{\ell^e} \right] d\xi = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$C^e = \int_{-1}^{1} \frac{\ell^e}{2} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \left[-\frac{1}{\ell^e} \ \ \frac{1}{\ell^e} \right] d\xi = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

AFTER ASSEMBLING Different elemetal matrices we get:

$$K = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \quad M = \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & 0 & 0 \\ \frac{1}{18} & \frac{2}{9} & \frac{1}{18} & 0 \\ 0 & \frac{1}{18} & \frac{2}{9} & \frac{1}{18} \\ 0 & 0 & \frac{1}{18} & \frac{1}{9} \end{bmatrix} \quad C = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$f = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}$$

Applying Dirich b.c. on $x=0$ and $x=1$, we obtain a reduce systerm.

$$\begin{bmatrix} 6+\frac{2}{9}\frac{1}{\Delta t} & \frac{1}{18}\frac{1}{\Delta t}-\frac{5}{2} \\ -\frac{7}{2}+\frac{1}{18}\frac{1}{\Delta t} & \frac{2}{9}\frac{1}{\Delta t}+6 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^{n+1} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \frac{1}{\Delta t}\begin{bmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^{n}$$

Initial guess $\underline{u}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $n=0$

The solution for the first time step. is $\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^1 = \begin{bmatrix} 6+\frac{2}{9}\frac{1}{\Delta t} & \frac{1}{18}\frac{1}{\Delta t}-\frac{5}{2} \\ -\frac{7}{2}+\frac{1}{18}\frac{1}{\Delta t} & \frac{2}{9}\frac{1}{\Delta t}+6 \end{bmatrix}^{-1}\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \frac{6\Delta t}{54\Delta t+5}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Operator splitting.

The operator splitting for this problem can be writlen as:

1° step $\frac{\Delta u_a^{n+1}}{\Delta t} + \mathcal{L}_a(u_a^{n+1}) = 0$ $\Delta u_a^{n+1} = u_a^{n+1} - u^n$

$\mathcal{L}_a(u) = \underline{a}\cdot\underline{\nabla} u$ (convective operator)

$$\left(\frac{M}{\Delta t} + C\right) u_a^{n+1} = \frac{M}{\Delta t} u^n$$

2° step $\frac{u^{n+1} - u_a^{n+1}}{\Delta t} + \mathcal{L}_k(u_a^{n+1}) = f^{n+1} + \frac{M}{\Delta t} u_a^{n+1}$ $\mathcal{L}_k(u) = \underline{\nabla}\cdot k\underline{\nabla} u$ (Diffusion operator)

$$\left(\frac{M}{\Delta t} + K\right) u^{n+1} = f^{n+1} + \frac{M}{\Delta t} u_a^{n+1}$$

Initial guess $\underline{u}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $n=1$

1° step: AFTER solving $\left(\frac{M}{\Delta t} + C\right) u_a^1 = 0$

$$u_a^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2° step: AFTER solving the epuation, taking into account u_a^1, then, weget:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{6}{54t+5}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It can be observe that monolithic solution is the same that one obtained with operator splitting.

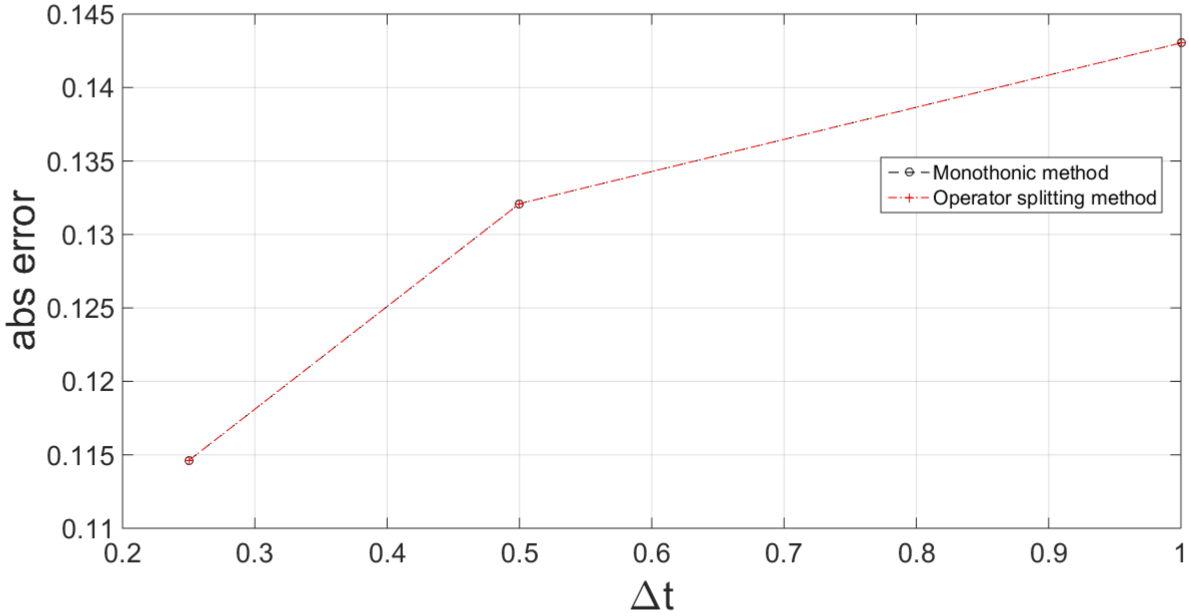


Figure 1: Error of Monolithic and Splitting method

6. FRACTIONAL STEP METHODS

1) optimal value of α prameter:

We are going to take the first and second equation and we will operate with them:

$$1^{\circ}\text{eq}\quad M\frac{1}{\delta t}(\hat{U}^{n+1}-U^{n})+K\hat{U}^{n+1}=f-G\hat{P}^{n+1}=\frac{M}{\delta t}\hat{U}^{n+1}-\frac{M}{\delta t}U^{n}+K\hat{U}^{n+1}=f-G\hat{P}^{n+1}$$

$$3^{\circ}\text{eq}\quad \frac{M}{\delta t}(U^{n+1}-\hat{U}^{n+1})+\alpha K(U^{n+1}-\hat{U}^{n+1})+G(P^{n+1}-\hat{P}^{n+1})=\frac{M}{\delta t}U^{n+1}-\frac{M}{\delta t}\hat{U}^{n+1}+\alpha KU^{n+1}-\alpha K\hat{U}^{n+1}$$
$$+GP^{n+1}-G\hat{P}^{n+1}=0$$

NOW, ADDING THE TWO EQUATION, WE GET:

$$\cancel{\frac{M}{\delta t}\hat{U}^{n+1}}-\frac{M}{\delta t}U^{n}+K\hat{U}^{n+1}+\frac{MU^{n+1}}{\delta t}-\cancel{\frac{M}{\delta t}\hat{U}^{n+1}}+\alpha KU^{n+1}-\alpha K\hat{U}^{n+1}+GP^{n+1}-\cancel{G\hat{P}^{n+1}}=f-\cancel{G\hat{P}^{n+1}}$$

$$\frac{M}{\delta t}(U^{n+1}-U^{n})+\alpha KU^{n+1}+(1-\alpha)K\hat{U}^{n+1}+GP^{n+1}=f$$

It can be seen if we choose $\alpha=1$, WE RECOVER the original momentum equation.

2) SOURCE TERM OF THE SCHEME:

In ORDER TO STABILIZE THE SOLUTION, the YOSIDA SCHEME ADDS A SMALL PERTURBATION IN THE CONTINUITY EQUATION. Tapically, the PERTURBED CONTINUITY EQUATION MAY TAKE ONE OF THE FOLLOWING FORMS:

$\nabla\cdot\underline{u}+\varepsilon\frac{\partial\psi}{\partial t}=0,\ \psi\big|_{t=0}=\psi_0$ ARTIFICIAL COMPRESSIBILITY

$\underline{\nabla}\cdot\underline{u}+\varepsilon\psi=0$ PENALIZATION

$\underline{\nabla}\cdot\underline{u}-\varepsilon\Delta\psi=0\quad \Delta\psi\cdot\underline{n}\big|_{\Gamma}=0$ Elliptic PRESSURE REGULARIZATION

THE PERTURBATION PARAMETER ε must be sufficiently large to have a significant regularization effect, but at the same time it should be kept as small as possible to minimizing the PERTURBATIONS ON the incompressibility equation. Since YOSIDA method guarentees the momentum equation but not the continuity equation, finally, the main SOURCE of ERROR of YOSIDA scheme is the unsatisfied continuity equation.

7. ALE FORMULATIONS

1_A)

The mapping between the mesh nodes χ and spatial coordinates x is:

$$\underline{x} = \phi(\underline{\chi}, t) = \underline{x}(\underline{\chi}, t)$$

And, the ALE coordinates are:

$$\gamma_{ALE} = \gamma(\underline{x}, t) = \gamma(\phi(\underline{\chi}, t), t) = [2(\chi + \alpha t), (y - \beta t)e^t, z]^T$$

1-B) WE CAN compute the velocity of the particle AS:

$$\underline{v}_p(\underline{X}; t) = \frac{\partial \underline{x}}{\partial t} = [\underline{X} e^t, e^t, 0]^T$$

and the velocity of the mesh as:

$$\underline{v}_{mesh} = \frac{\partial \underline{x}}{\partial t}(\chi; t) = [\alpha, -\beta, 0]^T$$

1-C) THE ALE DESCRIPTION OF THE MATERIAL TEMPORAL derivative of γ is:

$$\frac{d}{dt}\gamma_{ALE}(\chi(\underline{X}; t), t) = \frac{\partial \gamma_{ALE}}{\partial t} + (\underline{v}_p - \underline{v}_{mesh}) \cdot \underline{\nabla} \gamma(\underline{x}; t)$$

Where:

$$\frac{\partial \gamma_{ALE}}{\partial t} = [2\alpha, (y - \beta - \beta t)e^t, 0]^T \text{ ①} \quad \text{and} \quad \underline{v}_p - \underline{v}_{mesh} = [Xe^t - \alpha, e^t + \beta, 0]^T = [\chi + \alpha(t-1), e^t + \beta, 0]^T \text{ ②}$$

on the other hand:

$$\underline{\nabla}\gamma = \left[\frac{\partial \gamma_i}{\partial x_j}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ ③}$$

Replacing ①; ② and ③ in $\frac{d\gamma_{ALE}}{dt}$, we get:

$$\frac{\partial \gamma_{ALE}(\chi(\underline{X}; t), t)}{\partial t} = [2(\chi + \alpha t), e^t(y - \beta t + e^t), 0]^T$$

2) ALE FORMULATION Applied incompressible NAVIER-STOKES Equation.

THE ALE FORM OF THE NAVIER-STOKES EQUATION ARE READILY OBTAINED FROM THE CORRESPONDING well-KNOW EULERIAN FORM:

$$\rho \left.\frac{\partial v}{\partial t}\right|_x + \rho(v \cdot \underline{\nabla} v) = \underline{\nabla} \cdot \underline{\sigma} + \rho \underline{b} \quad \text{(MOMENTUM in EULERIAN FORM)}$$

$$\left.\frac{\partial \rho}{\partial t}\right|_x + v \cdot \underline{\nabla} \rho = -\rho \underline{\nabla} \cdot \underline{v} = 0 \quad \text{(MASS in EULERIAN FORM)}$$

All one has to do to obtain the ALE FORM of the ABOVE Equations is to REPLACE in the convective terms, the material velocity v with the convective velocity

$$\underline{C} = \underline{v}_p - \underline{v}_{mesh}$$

$$\rho \frac{\partial \underline{v}}{\partial t}\Big|_{X} + \rho(\underline{c}\cdot\underline{\nabla})\underline{v} = \underline{\nabla}\cdot\underline{\sigma} + \rho\underline{b}$$ written in ALE FORM.

EVALUATED AT MESH NODES

$$\frac{\partial \rho}{\partial t}\Big|_{X} + \underline{c}\cdot\underline{\nabla}\rho = -\rho\,\underline{\nabla}\cdot\underline{v}=0$$ written in ALE FORM.

EVALUATED A MESH NODES

It is important tonote that the right-hand side of the equation written above are expresed in CLASSICAL EULERIAN form, while the arbitrary motion of the computational mesh is only REFLECTED in the left-hand side. ON THE OTHER HAND, THE TIME DISCRETIZATION CAN BE DONE WITH Any DIFFERENCE methods in term of the mesh NODES. The temporal Derivative is evaluated AS the Difference from n to n+1 at a moving NODES.

3) Bibliographical research:

THE MAJority of modern ALE computer codes are based on either finite volume or finite element spatial discretization, the former being popular in the fluiD mechanics AreA, the latter being generally preferred in SOLID and structural mechanics or for Application of the element-free Galerkin method to dynamic fracture problems. One of the main adventages of the ALE formulation is that it represents a very versatile combination of the classical LAgrangiaL and EULERIAN Descriptions. However, the computer implementation of the ALE technique requires the formulation of a mesh-update procedure that assigns mesh-node velocities or displacements at each station of a calculation. The mesh update strategy can be chosen by the user.

Two basic mesh-update strategies may be identified. The geometrical concept of mesh regularization can be exploited to keep the computational mesh as regular as possible and to avoid mesh entanglement during the calculation. On the other hand, if the ALE approch is used as mesh-adaptation technique, for instance, to concentrate elements in zone of steep solution gradient, a suitable indication of the error is required as a basic input to the re-mesh algorithm.

The mesh Regularization

The objective consists in keeping the computational mesh as regular as possible during the whole calculation, thereby avoiding excessive distortions and squeezing of the compution zones and preventing mesh entanglement. This procedure decreases the numerical error due to the mesh distortion. Mesh regularization requires that updated nodal coordinates be specified at each station of a calculation, either through step displacements, or from current mesh velocity $\underline{v}_{mesh}$. Usually, in fluid flows, the mesh velocity is interpolated, and in SOLID problems, the mesh displacement is directly interpolated the interaction problem between a rigid body and a viscous fluid studied by [4] falls in this category. Similary, the crack propagation problems discussed by [2] and [3], where the crack path is known a priori, also allow the use of this kind of mesh-update procedure.

MESH ADAPTATION

When ALE Description is used as an adaptive mesh, the objective is to optimize the computational mesh to achive to improved accuracy, possibly at low computational cost (the total number elements in a mesh remains unchanged throughout the computational, as well as the element connectivity). The ALE algorithm then includes an indicator of the error, and the mesh is modified to obtain an equi-distribution of the error over the entire computational Domain. the re-mesh indicator can, for instance, be made a function of the average or the jump of a certain state variable. The ALE technique can nevertheless be coupled with traditional mesh-refinement procedures, such as h-adaptivity, to further enhance accuracy through the selective addition of new degrees of freedom. [1]

REFERENCE

[1] "Arbitrary Lagrangian – EULERIAN METHODS". J. Donea, Antonio Huerta, J. Ph. Ponthot and A. RODRIGUEZ - FERRAN.

[2] "ELASTODINAMIC FORMULATION of the EULERIÁN-LAGRANGIAN Kinematic Description". Kohn HM and Heber RB.

[3] "Dynamic crack PROPAGATION ANALYSIS USING EULERIAN-LAGRANGIAN Kinematic Description" Koh Hm et al.

[4] "VISCOUS Flow structure interACTION" HUERTA A. and Liu WK.

8. FLUID-STRUCTURE INTERACTION

1_ Added mass effect:

Occure when the fluid is similar to the solid density; For instance, body tissues vs water, the partitioned schemes do not work properly.
The Added mass operator describes, how the prediction of the interface acceleration relates to the new interface force for the structure problem. It acts as additional mass on the degree of freedom on the interface.
Exist numerical techniques such as Aitken relaxation scheme, steepest-descent method and using robin-robin boundary conditions, that can mitigate this problem.

2 _ 1D HEAT TRANSFER PROBLEM:

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f \quad (f = k = 1) \\ u = 0 \quad \text{on } \partial\Omega \ \forall t \in \mathbb{R}^+ \end{cases}$$

AFTER DISCRETIZING in time (BDF1) and space, we get:

$$\left(\frac{M}{\Delta t} + K\right) U^{n+1} = f + \frac{M}{\Delta t} U^n$$

Now, we split the Domain in two subdomains:

• subdomain Ω_1

$$\begin{cases} \left[\frac{M_1}{\Delta t} + K_1\right] U_1^{n+1,k} = f_1 + \frac{M_1}{\Delta t} U_1^n \quad \text{in } \Omega_1 \\ \frac{du_{\Gamma_{12}}^{n+1,k}}{dx} = -\frac{du_{\Gamma_{21}}^{n+1,k-1}}{dx} \quad \text{on } \Gamma_{12} \\ u^{n+1,k} = 0 \end{cases}$$

Then, we are going to calculate the Dirichlet value for the problem in subdomain 2 using Aitken's relaxation scheme.

$$\omega = \frac{u_{\Gamma_{21}}^{n+1,k-2} - u_{\Gamma_{21}}^{n+1,i-1}}{u_{\Gamma_{21}}^{n+1,i-2} - u_{\Gamma_{21}}^{n+1,i-1} + u_{\Gamma_{12}}^{n+1,i} - u_{\Gamma_{12}}^{n+1,i-1}}$$

• subdomain Ω_2 : Dirichlet problem

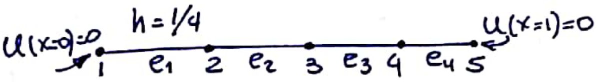
$$\begin{cases} \left(\frac{M_2}{\Delta t} + K_2\right) U_2^{n+1,k} = f_2 + \frac{M_2}{\Delta t} U_2^n \quad \text{in } \Omega_2 \\ u_{\Gamma_{21}}^{n+1,k} = \omega u_{\Gamma_{12}}^{n+1,k} + (1-\omega) u_{\Gamma_{21}}^{n+1,i-1} \quad \text{on } \Gamma_{21} \\ u_2^{n+1,k} = 0 \quad \text{on } \partial\Omega_2 \setminus \Gamma_{21} \end{cases}$$

- It can be seen that the relaxation parameter(w) needs more that two iteration, otherwise it remains constant. After two iteration, AitKen's method can be used in order to heve a better convergence rate.

3- Monolitic scheme:

We consider the following parameter and mesh size:

$k=1$; $h=\frac{1}{4}$; 4 linear elements.



- Elemental stiffness matrix K^e: $\Rightarrow K^e = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

- Elemental Mass matrix $M^e \Rightarrow M^e = h\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix}$

- Assembling:

$$K = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix}$$

- Using Lagrange multipliers:

$$L.u = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The new Algebraic system of equation to be solved is:

$$\tilde{A}\,\tilde{U}^{n+1} = \tilde{f} \Rightarrow \begin{bmatrix} A & L^T \\ L & 0 \end{bmatrix}\begin{bmatrix} U^{n+1} \\ \lambda \end{bmatrix} = \begin{bmatrix} \hat{f} \\ b \end{bmatrix}$$

Where $A = \frac{M}{\Delta t} + K$

$$\hat{f} = f + \frac{M}{\Delta t} U^n$$

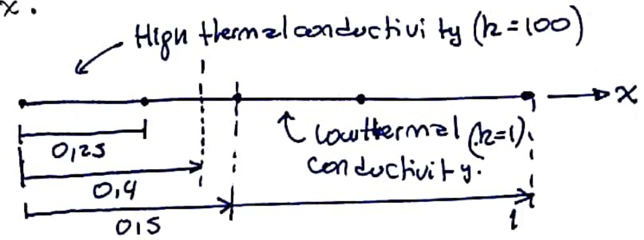
- The condition number of any matrix is defined as $k_A = \|A\|.\|A^{-1}\|$.

So, the condition number of the resulting matrix (After using matlab). $k_{\tilde{A}} = 36{,}87$.

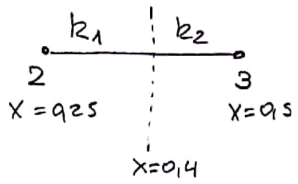
4-

The Form of the Algebraic system of equation is the same. But, we have to correct the stiffness matrix.

Now, our Domain is.



1° we have to take into account that for element 2 we have two types of thermal conductivities.



So the component of the elemtal stiffness matrix for the element two will be:

$$K_{22}^{(2)}=K_{11}^{(2)}=k_1\int_{0.25}^{0.4}\left(-\frac{x}{h}\right)\left(-\frac{x}{h}\right)dx+k_2\int_{0.25}^{0.4}\left(-\frac{x}{h}\right)\left(-\frac{x}{h}\right)dx=242.4$$

$$K_{21}^{(2)}=K_{12}^{(2)}=k_1\int_{0.25}^{0.4}\left(-\frac{x}{h}\right)\left(\frac{x}{h}\right)dx+k_2\int_{0.25}^{0.4}\left(-\frac{x}{h}\right)\left(\frac{x}{h}\right)dx=-242.4$$

The elemtal stiffness matrix for elemnt 1 will be the same as problem 3, but multiply by $k_1=400$.

- Assembing:

$$K_{glob}=\begin{bmatrix}400 & -400 & 0 & 0 & 0\\ -400 & 642.4 & -242.4 & 0 & 0\\ 0 & -242.4 & 246.4 & -4 & 0\\ 0 & 0 & -4 & 8 & -4\\ 0 & 0 & 0 & -4 & 4\end{bmatrix}\quad M_{glob}=\begin{bmatrix}1/12 & 1/24 & 0 & 0 & 0\\ 1/24 & 1/6 & 1/24 & 0 & 0\\ 0 & 1/24 & 1/6 & 1/24 & 0\\ 0 & 0 & 1/24 & 1/6 & 1/24\\ 0 & 0 & 0 & 1/24 & 1/12\end{bmatrix}$$

The Algebraic system to be solved and After Applying Dirichel Boundary condition.

$$\left(\frac{M}{\Delta t}+K\right)U^{n+1}=f+\frac{M}{\Delta t}U^{n}$$