Homeworks
Coupled Problems

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1 Transmission Conditions

1.1 Problem 1

Given the Principle of Virtual Work (PVW) for the deflection of a Euler-Bernouilli beam (1) clamped at both ends:

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f \quad \forall \delta v \quad \begin{aligned} \frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0 \\ \delta v(0) = \delta v(L) = 0 \end{aligned} \tag{1}$$

1. Space of functions where v and δv must belong:

   The right hand side of has to be integrable, so this implies that δv has to be at least integrable:

   $$\int_0^L \delta v f < \infty \longleftrightarrow \delta v \in L^2 \tag{2}$$

   So δv has to be at least in L^2.
   The integral on the left of has also to be bounded, so to be in the space L^2.

   $$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} < \infty \qquad \in L^2$$

   So the second derivative of δv ($\frac{d^2\delta v}{dx^2}$) and v ($\frac{d^2 v}{dx^2}$)also has to be bounded ($\in L^2$). The space of functions with continuous second derivatives is H^2, so:

   $$\delta v, v \in H^2$$

2. If $\Omega = [0, L] = [0, P] \cup (P, L]$, obtain the transmission conditions at P implied by regularity requirements.

We have proven that the deflection of the beam has to be in H^2, therefore $v, \frac{dv}{dx}, \frac{d^2v}{dx^2}$ must be in L^2. The imposition of regularity of the deflection of the beam v is derived in the following way considering a regularised function v^ε for the deflection and dv^ε for the first derivative connecting two points separated a distance ε across the boundary Γ_P of the partition of Ω. In this way the deflection v can be seen as the limit of the regularised function when the separation between points goes to zero, and its first derivative similarly:

$$v = \lim_{\varepsilon \to 0} v^\varepsilon \qquad \frac{dv}{dx} \mid_P = \lim_{\varepsilon \to 0} \frac{dv^\varepsilon}{dx} \mid_P \tag{3}$$

The integral across the interface Γ_P from $P - a$ to $P + a$ of the first derivative of the unknown can be slitted in three sections as:

$$\int_{P-a}^{P+a} \frac{du^\varepsilon}{dx} = \int_{P-a}^{P-\frac{\varepsilon}{2}} \frac{dv}{dx} + \int_{P-\frac{\varepsilon}{2}}^{P+\frac{\varepsilon}{2}} \frac{dv^\varepsilon}{dx} + \int_{P+\frac{\varepsilon}{2}}^{P+a} \frac{dv}{dx}$$

and using the definition of the derivative and recalling (3):

$$\int_{P-a}^{P+a} \frac{dv^\varepsilon}{dx} = \int_{P-a}^{P-\frac{\varepsilon}{2}} \left(\frac{dv}{dx} + \varepsilon \left[\frac{v(P+\frac{\varepsilon}{2}) - v(P-\frac{\varepsilon}{2})}{\varepsilon} \right] \right) + \int_{P+\frac{\varepsilon}{2}}^{P+a} \frac{dv}{dx}$$

but since we want $\varepsilon \to 0$ this expression can be finally understood as:

$$\int_{P-a}^{P+a} \frac{dv^\varepsilon}{dx} = \int_{P-a}^{P-\frac{\varepsilon}{2}} \left(\frac{dv}{dx} + \left[v(P^+) - v(P^-) \right] \right) + \int_{P+\frac{\varepsilon}{2}}^{P+a} \frac{dv}{dx}$$

Till now we have been considering v to be discontinuous across the boundary Γ_P, but if we want v to be regular the integral has to be in L^2, what means imposing (2):

$$\int_{P-a}^{P+a} \left(\frac{dv^\varepsilon}{dx} \right)^2 = \int_{P-a}^{P-\frac{\varepsilon}{2}} \left(\left(\frac{dv}{dx}^2 + \varepsilon \left[\frac{(v(P+\frac{\varepsilon}{2}) - v(P-\frac{\varepsilon}{2}))^2}{\varepsilon^2} \right] \right) + \int_{P+\frac{\varepsilon}{2}}^{P+a} \left(\frac{dv}{dx} \right)^2$$

Clearly the term with ε^2 in the denominator when $\varepsilon \to 0$ will go to infinity. Therefore the integral only can be in L^2 if this term vanishes, what happens when (5):

$$(v(P + \frac{\varepsilon}{2}) - v(P - \frac{\varepsilon}{2})) = v(P^+) - v(P^-) = v_{\Gamma_P} = 0 \tag{4}$$

Which is the first transmission condition that states the continuity of the unknown (de deflection v in this case), or equivalently that the jump $v_{\Gamma_P}$of the unknown across the interface of two subdomains has to be zero.

Proceeding in the same way but now considering dv^ε instead of v we arrive to:

$$\int_{P-a}^{P+a} \left(\frac{d^2v^\varepsilon}{dx^2} \right)^2 = \int_{P-a}^{P-\frac{\varepsilon}{2}} \left(\left(\frac{d^2v}{dx^2} + \varepsilon \left[\frac{(dv(P+\frac{\varepsilon}{2}) - dv(P-\frac{\varepsilon}{2}))^2}{\varepsilon^2} \right] \right) + \int_{P+\frac{\varepsilon}{2}}^{P+a} \left(\frac{d^2v}{dx^2} \right)^2$$

what implies that the first derivative of the deflection dv that corresponds to the rotation of the beam must be equal in the interface Γ_P:

$$(dv(P+\frac{\varepsilon}{2})-dv(P-\frac{\varepsilon}{2}))=dv(P^+)-dv(P^-)=dv_{\Gamma_P}=0 \tag{5}$$

3. Obtain the transmission conditions at P that follow by imposing in the PVW that the integral is additive.

The variational form (1) can be split into the integration by subdomains, for a general case where EI could be different in the two subsets:

$$(EI)_1\int_0^{P-}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}+(EI)_2\int_{P-}^{L}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}=\int_0^L\delta v f$$

And these integrals can be split as an integral in the contour of the subdomain and another in the interior using integration by parts. For example proceeding with the first one:

$$(EI)_1\int_0^{P-}\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}=(EI)_1\left[\left(\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\right)_0^{P-}-\int_0^{P-}\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]=$$

$$=(EI)_1\left[\left(\frac{d\delta v}{dx}\frac{d^2v(P-)}{dx^2}\right)-\int_0^{P-}\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]$$

In this equation the second term is exactly the variational form resulting from applying once to the strong form integration by parts. Therefore has to be equal to the strong form of problem, what means that the term in the interface when adding the two contributions of both subdomains we end having:

$$(EI)_1\left[\left(\frac{d\delta v}{dx}\frac{d^2v(P-)}{dx^2}\right)-\int_0^{P-}\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]+(EI)_2\left[\left(\frac{d\delta v}{dx}\frac{-d^2v(P+)}{dx^2}\right)-\int_{P-}^{L}\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]$$

From where the third transmission condition is deduced:

$$(EI)_1\left(\frac{d^2v(P-)}{dx^2}\right)-(EI)_2\left(\frac{d^2v(P+)}{dx^2}\right)=0 \tag{6}$$

Repeating the same procedure but with the first variational form of the strong form:

$$(EI)_1\left[\left(\delta v\frac{d^3v(P-)}{dx^3}\right)-\int_0^{P-}\delta v\frac{d^4v}{dx^4}\right]+(EI)_2\left[\left(\delta v\frac{-d^3v(P+)}{dx^3}\right)-\int_{P-}^{L}\delta v\frac{d^4v}{dx^4}\right]$$

The integral corresponds to the strong form of the problem integrated against the test function, so has to be equal to the strong form. Therefore we end up with the fourth transmission condition which is:

$$(EI)_1\left(\frac{d^3v(P-)}{dx^3}\right)-(EI)_2\left(\frac{d^3v(P+)}{dx^3}\right)=0 \tag{7}$$

1.2 Problem 2

The Maxwell problem consists in finding a vector field $u : \Omega \longrightarrow R^3$ such that:

$$\begin{aligned} \nu\nabla \times \nabla \times u = f \quad & \in \Omega \\ \nabla \cdot u = 0 \quad & \in \Omega \\ n \times u = 0 \quad & on \quad \partial\Omega \end{aligned} \tag{8}$$

where $\nu >$ f is a divergence free force field and n the unit external normal. Equation $\nabla \cdot v = 0$ is in fact redundant.

1. Write a variational statement of the problem. Postulate the space of functions where u must belong. Justify the answer.

   To derive the variational form we have to multiply both sides by a test function $v = 0$ on Γ_N.

   $$\int_\Omega (\nu\nabla \times \nabla \times u) \cdot v = \int_\Omega fv$$

   Then integrating by parts the left hand side we obtain:

   $$\int_\Omega \nu\nabla \times u \cdot \nabla \times v - \int_{\partial\Omega} (\nu\nabla \times u \times n) \cdot v = \int_\Omega fv$$

   and since we have imposed that the tangential component of u is null $u \times n = n \times u = 0$ on the boundary $\partial\Omega$, finally the weak variational form of the Maxwell's equation is:

   $$\int_\Omega \nu\nabla \times u \cdot \nabla \times v = \int_\Omega fv$$

   Both functions u and v appear affected by the curl operator inside the integral. Therefore in order to have a finite integral this functions must belong to the space $H(curl)$ which is defined as the Hilbert space of square integral functions and also square integrable curl:

   $$u, v \in H(curl) := u, v \in L^2, \nabla \times (u, v) \in L^2$$

2. If Γ is a surface that intersect Ω, obtain the transmission conditions across Γ implied by regularity requirements.

   Due to regularity conditions that u must satisfies to be in the space $H(curl)$. The first transmission condition is deduced from the fact that a function u in 3D to be in H^1 cannot be discontinuous across a surface Γ. In his way the first transmission condition for Maxwell's equation is:

   $$v_\Gamma = 0$$

3. Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

   Operating in the variational form for a subdomain Ω_1, we get:

   $$\int_{\Omega 1} \nu\nabla\times u\cdot\nabla\times v = \int_{\Omega 1}(\nu\nabla\times\nabla\times u)\cdot v + \int_{\partial\Omega 12}(\nu\nabla\times u\times n)\cdot v = \int_{\Omega 1} fv$$

   But since the first term of the integral is equal to the strong form integrates against a test function, it has to be equal to the variational form. Therefore when adding the integrals of the 2 subdomains, the next condition must hold:

   $$\int_{\partial\Omega 12}(\nu\nabla\times u\times n)\cdot v - \int_{\partial\Omega 21}(\nu\nabla\times u\times n)\cdot v = 0$$

1.3 Problem 3

The Navier equations for an elastic material can be written in three different ways:

$$\begin{aligned} -2\mu\nabla\cdot(\varepsilon(u)) - \lambda\nabla(\nabla\cdot u) &= \rho b \\ -\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) &= \rho b \\ \mu\nabla\times(\nabla\times u) - (\lambda-2\mu)\nabla(\nabla\cdot u) &= \rho b \end{aligned} \tag{9}$$

where u is the displacement field, $\varepsilon(u)$ the symmetric part of ∇u, λ and μ the Lamé coefficients, ρ the density of the material and b the body forces. Let us assume that $u = 0$ on $\partial\Omega$.

1. Write down the variational form of the previous equations in the appropriate functional spaces.

   The variational form of the first expression of the Navier equation is:

   $$2\mu\int_\Omega \nabla^S u : \nabla v - 2\mu\int_{\partial\Omega}\nabla^S u\cdot v - \lambda\int_\Omega \nabla u : \nabla v - \lambda\int_\Omega \nabla\cdot(\nabla u\cdot v) = \int_\Omega \rho b\cdot v$$
   $$2\mu\int_\Omega \nabla^S u : \nabla v - 2\mu\int_{\partial\Omega}\nabla^S u\cdot v - \lambda\int_\Omega \nabla u : \nabla v - \lambda\int_{\partial\Omega}(n\cdot\nabla u)\cdot v = \int_\Omega \rho b\cdot v$$

   In this case both vector fields $u, v \in H^1, R^3$ because they appear in the integrals affected by a gradient operator.

   The second variational form would be:

   $$\mu\int_\Omega \nabla u : \nabla v - \mu\int_{\partial\Omega}\nabla u\cdot v - (\lambda+\mu)\int_\Omega \nabla u : \nabla v - (\lambda+\mu)\int_{\partial\Omega}(n\cdot\nabla u)\cdot v = \int_\Omega \rho b\cdot v$$

   Again in this case both vector fields $u, v \in H^1, R^3$ because they appear in the integrals affected by a gradient operator.

Finally the variational form of the third expression of the Navier equation is:

$$\mu \int_{\Omega} \nabla \times u \cdot \nabla \times v - \mu \int_{\Omega} (\nabla \times u \times n) \cdot v - (\lambda - 2\mu) \int_{\Omega} \nabla u : \nabla v - (\lambda - 2\mu) \int_{\partial\Omega} (n \cdot \nabla u) \cdot v = \int_{\Omega} \rho b \cdot v$$

In this case the vector fields $u, v \in H(curl), R^3$ in order to have bounded integrals.

2. If Γ is a surface that intersect Ω, obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

2 Domain decomposition methods

2.1 Problem 1

Consider Problem 1 of Transmission condition section. Let $[0, L] = [0, L_1][[L_2; L]$, with $L_2 < L1$.

1. Write down an iteration-by-subdomain scheme based on a Schwarz additive domain decomposition method.

   Schwarz algorithm is a parallel iterative iteration-by-subdomain algorithm because there is no yielding between subdomains. Consists in solving iteratively the weak form of the problem in the overlapping subdomains and impose boundary conditions and transmission conditions coming from the previous iteration in the other subdomain. For the case of the Euler-Bernouilli beam theory, from the split of the integral in the two overlapping subdomains proposed:

$$\begin{array}{rlrl}
 & Subdomain \quad \Omega_1 & & Subdomain \quad \Omega_2 \\
(EI)_1 \int_0^{L_1} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^{L_1} \delta v f & & (EI)_2 \int_{L_2}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_{L_2}^{L} \delta v f & \\
v(0) = 0 & x = 0 & v(0) = 0 & x = L \\
\frac{dv(0)}{dx} = 0 & x = 0 & \frac{dv(0)}{dx} = 0 & x = L \\
v_1^{(k)} = v_2^{(k-1)} & \Gamma_{12} & v_2^{(k)} = v_1^{(k-1)} & \Gamma_{21} \\
\frac{dv_1}{dx}^{(k)} = \frac{dv_2}{dx}^{(k-1)} & \Gamma_{12} & \frac{dv_2}{dx}^{(k)} = \frac{dv_1}{dx}^{(k-1)} & \Gamma_{21} \\
(EI)_1 \frac{d^2 v_1}{dx^2}^{(k)} = (EI)_2 \frac{dv_2^2}{dx^2}^{(k-1)} & \Gamma_{12} & (EI)_2 \frac{d^2 v_2}{dx^2}^{(k)} = (EI)_1 \frac{dv_1^2}{dx^2}^{(k-1)} & \Gamma_{21} \\
(EI)_1 \frac{d^3 v_1}{dx^3}^{(k)} = (EI)_2 \frac{dv_2^3}{dx^3}^{(k-1)} & \Gamma_{12} & (EI)_2 \frac{d^3 v_2}{dx^3}^{(k)} = (EI)_1 \frac{dv_1^3}{dx^3}^{(k-1)} & \Gamma_{21}
\end{array}$$

2. Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

   The discretization using Finite Elements is done with the Galerkin formulation, this is selecting the test function δv to be the same as the shape function. In this way the variational form of the deflection of the beam is expressed as the system of equations:

$$
\begin{aligned}
A\mathbf{u} &= b \\
A &= \sum_{i,j}^{n} \frac{d^2\delta v_j}{dx^2}\frac{d^2\delta v_i}{dx^2} \qquad (i,j) = 1, n \\
\mathbf{u} &= [u_j] \quad j = 1, n \\
b &= \sum_{j}^{n} \delta v_j f
\end{aligned}
$$

   Once the discretization is done, the matrix version of the Schwarz algorithm consists in order the vector of unknowns as interior nodes at the top and interface ones below, and create the matrix by blocks so the problem ends being:

$$
\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_i^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} b_i \\ b_\Gamma \end{bmatrix}
$$

   where $u_i^{(k)}$ is the solution in the interior nodes and $u_\Gamma^{(k)}$ the one in the overlapping interface region that is responsible of the transmission conditions. The solution in an iterative by subdomains approach leads to:

$$
\begin{aligned}
A_{11}u_1^{(k)} &= b_1 - A_{1\Gamma}u_\Gamma^{(k-1)} \\
A_{22}u_2^{(k)} &= b_2 - A_{2\Gamma}u_\Gamma^{(k-1)}
\end{aligned}
$$

2.2 Problem 2

Consider Problem 2 of Section 1. Let Γ be a surface that intersects Ω.

1. Write down an iteration-by-subdomain scheme based on the Dirichlet-Neumann coupling.

   The Dirichlet-Neumann integration by subdomains is stated in the following way for the Maxwell equation:

$$\begin{aligned}
&\textit{Subdomain} \quad \Omega_1\\
&\int_{\Omega 1} \nu\nabla \times u \cdot \nabla \times v = \int_{\Omega 1} fv\\
&u^{(k)} = \bar{u} \quad \Omega_1\\
&\int_{\Gamma 12} \left(\nu\nabla \times u^{(k)} \times n\right) \cdot v =\\
&= \int_{\Gamma 21} \left(\nu\nabla \times u^{(k-1)} \times n\right) \cdot v \quad \Gamma_{12}
\end{aligned}
\qquad
\begin{aligned}
&\textit{Subdomain} \quad \Omega_2\\
&\int_{\Omega 2} \nu\nabla \times u \cdot \nabla \times v = \int_{\Omega 2} fv u_2^{(k)} = u_1^{(l)}\\
&u^{(k)} = \bar{u} \quad \Omega_2\\
&\Gamma_{21}
\end{aligned}$$

if $l = k - 1$, parallel scheme (Jacobi) and if $l = k$, serialised scheme (Gauss-Seidel).

2. Obtain the expression of the Steklov-Poincaré operator of the problem.

   The Steklov-Poincaré operator in the Maxwell problem would look like:

$$S : H^{\frac{1}{2}}(\Gamma_{12}) \longrightarrow H^{\frac{-1}{2}}(\Gamma_{12})$$

$$\varphi \longrightarrow \int_{\partial\Omega 12} (\nu\nabla \times \tilde{u} \times n) \cdot v - \int_{\partial\Omega 21} (\nu\nabla \times \tilde{u} \times n) \cdot v$$

3. Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

   The discretization using finite elements (Galerkin approximation) consists in replace in the variational formulation:

$$\mathbf{u} = \sum_{j=1}^{n} v_j u_j$$

   In this way the matrix and force vector of the system to solve are:

$$A = \sum_{i,j} \nu\nabla \times v_j \cdot \nabla \times v_i \quad (i,j) = 1, n$$

$$b = \sum_{j}^{n} \delta v_j f$$

   So the Dirichlet-Neumann iterative algorithm in matrix form appears to be:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_\Gamma - A_{\Gamma 2} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22} u_2^{(k)} = b_\Gamma - A_{2\Gamma} u_\Gamma^{(l)}$$

   Where A_{ii} solves the interior problem in the subdomain and $u_\Gamma^{(l)}$ is responsible of communicating the transmission conditions between subdomains.

2.3 Probelm 3

Consider the problem of finding $u : \Omega \to R$ such that

$$\begin{aligned} -f\Delta u &= f \quad in\Omega \\ u &= 0 \quad on\partial\Omega \end{aligned}$$

where $k > 0$. Let Γ be a surface crossing Ω.

1. Write down an iteration-by-subdomain scheme based on the Dirichlet-Robin coupling.

   The weak form of the Poisson's problem is well known ad has the expression:

   $$-k \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv$$

   And the Dirichlet-Robin transmission conditions on Γ obtained for this problem are:

   $$\begin{aligned} u_{\Gamma_{12}} &= u_{\Gamma_{21}} \\ \nabla u_1 + \gamma_2 u_1 &= \nabla u_2 + \gamma_2 u_2 \end{aligned}$$

   So the Dirichlet-Robin iteration-by-subdomains scheme is:

   $$\begin{array}{cc} Subdomain \quad \Omega_1 & Subdomain \quad \Omega_2 \\ -k \int_{\Omega_1} \nabla u_1 \cdot \nabla v = \int_{\Omega_1} fv & -k \int_{\Omega_2} \nabla u_2 \cdot \nabla v = \int_{\Omega_2} fv \\ u_1^{(k+1)} = u_2^{(k)} & \nabla u_2^{(k+1)} + \gamma_2 u_2^{(k+1)} = \nabla u_1^{(l)} + \gamma_1 u_1^{(l)} \end{array}$$

   if $l = k - 1$ the problem is parallel (Jacobi) and if $l = k$ is serialised (Gauss-Seidel).

2. Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

   The matrix version of the is for subdomain Ω_1:

   $$\begin{aligned} &Subdomain \quad \Omega_1 \\ u_1^{k+1} &= A_{11}^{-1}(f_1 - A_{1\Gamma} u_\Gamma^k) \\ &Subdomain \quad \Omega_2 \end{aligned}$$

   $$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_2^{k+1} \\ u_\Gamma^{k+1} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_\Gamma - A_{\Gamma 2} u_1^{(k)} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) u_\Gamma^{(k)} \end{bmatrix}$$

3. Obtain the Schur complement as discrete version of the Steklov-Poincaré operator.

The Steklov-Poincaré operator for this problem takes the form:

$$S : H^{\frac{1}{2}}(\Gamma_{12}) \longrightarrow H^{\frac{-1}{2}}(\Gamma_{12})$$
$$-\nabla \bar{u}_1 + \gamma_2 u_1 + \nabla \bar{u}_2 + \gamma_2 u_2$$

And from th Steklov-Poincaré operator we obtain the discrete version that corresponds to the Schur complement:

$$S_1 = A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$$
$$S_2 = A_{\Gamma\Gamma} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) A_{22}^{-1} (A_{2\Gamma} + A_{\Gamma\Gamma})$$
$$S = A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) A_{22}^{-1} (A_{2\Gamma} + A_{\Gamma\Gamma})$$

4. Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section 1.

   Considering a Gauss-Seidel iterative scheme:

$$u_1^k = A_{11}^{-1}(f_1 - A_{1\Gamma} u_\Gamma^k)$$
$$u_2^k = A_{22}^{-1}(f_2 - A_{2\Gamma} u_\Gamma^k)$$

3 Coupling of heterogeneous problems

3.1 Problem 1

Consider the beam described in Problem 1 of Section 1. Apart from being clamped at $x = 0$ and $x = L$, the beam is supported on an elastic wall that occupies the square $[0, L] \times [-L, 0]$, where $y = 0$ corresponds to the beam axis. The wall is clamped everywhere except on the upper wall, where the beam is. The wall displacements in the x- and y-directions are u and v, respectively, and the elastic properties E (Young modulus) and ν (Poisson's coefficient). No loads are applied on the wall, except for those coming from the beam.

1. Write down the equations in the wall assuming a plane stress behavior.

   The plane stress theory is represented by an stress state of the form:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   with the constitutive equation written in Voigt's notation:

$$\begin{aligned} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} \\ \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{bmatrix} \end{aligned} \tag{10}$$

And the conservation law in the wall considering no external forces nor body ones applied to it excepts for the ones of the beam, and neglecting time variations:

$$\nabla \cdot \sigma = 0$$

2. Write down the equations for the beam modified because of the presence of the wall.

The vertical deflection of the beam v now is subjected to the impediment of the wall, that exerts an opposite force to the distributed load f that makes the external load on the beam change. The new expression has the form:

$$EI\frac{d^4v}{dx^4} = f$$

$$\begin{aligned} f &= f_{ext} - \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_y}{\partial y}\right) = \\ &= f_{ext} - \frac{E}{1-\nu^2}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)\right] \end{aligned}$$

3. Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at $y = 0$.

The first transmission condition in the wall is that the if the beam deforms downwards as it will do under the vertical distributed load f there has to be contact, so both vertical displacements have to be the same:

$$v(y=0)_{beam} = v(y=0)_{wall}$$

The transmission conditions on the normal component on the tractions is that at the interface ($y = 0$) the tractions computed with the constitutive law using the displacements of the beam:

$$\sigma \cdot n = \sigma(u_{beam}) \cdot n \quad on, y = 0$$

4. *Suggest* transmission conditions for u and the tangent component of the traction on the wall at $y = 0$. Discuss the implications if this component is not assumed to be zero.

A possible transmission condition for u would be that the x-displacement of the solid at the boundary $y = 0$ follows linear elasticity theory under the action of friction forces between the two materials.

On the other hand the tangential component of the stress τ_{xy} on the interface can be computed using its expression in (10) with the x and y displacement given by the Dirichlet transmission conditions. However if this component is not set to zero the angular momentum on the linear elastic solid might be unbalanced.

3.2 Problem 3

Let S_D and S_S be the Dirichlet-to-Neumann operators for the Darcy and the Stokes problems, respectively. The Steklov-Poincaré equation can be written as:

$$S_S(\lambda) = S_D(\lambda)$$

where λ is the normal velocity on Γ in the interface between the Darcy and the Stokes regions.

1. Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

   The equations of Stokes and Darcy's problems are:

$$\begin{aligned} \textit{Stokes}: &\qquad \textit{Darcy}: \\ -\nu\Delta u_S + \nabla p_S = f &\qquad u_D + K\nabla\varphi = 0 \\ \nabla \cdot u_S = 0 &\qquad \nabla \cdot u_D = 0 \end{aligned}$$

   Discretizing with finite elements the weak form of these equations with Galerkin approximation, we end up with the following matrix:

$$-\sum_{ij}\int_{\Omega^e}(\nu\nabla(\delta u_S)_i : \nabla(\delta u_S)_j - (p_S)_i\nabla\cdot(\delta u_S)_j)$$
$$-\sum_{j}\int_{\partial\Omega}((\delta u_S)_i\cdot(n\cdot(-p_S I + \nu\nabla(\delta u_S)_j)(u_S)_j))$$

   Since velocity and pressure can be in very different spaces, it might be necessary to introduce mixed FEM formulation for velocity and pressure. Here after the pressure discretisation will be:

$$p_S = \sum_i p_i \delta p_i$$

   On the other hand the Dacy's flow equation is discretized with finite elements as:

$$\sum_{ij}\int_{\Omega^e}\left((\delta u_D)_i\cdot K^{-1}\cdot(\delta u_D)_j u_D - \varphi_i\nabla\cdot(\delta u_D)_j\right)$$
$$\sum_j\int_{\partial\Omega}(\varphi n\cdot(\delta u_D)_j$$

   Since the integrals are additive, the matrices of the Stokes and Darcy flow can be separated in a velocity and pressure terms.

The differential form of the Steklov-Poincaré operators are:

$$S_S(\lambda) = (p_S - n \cdot \nu \nabla u_S \cdot n) \mid_\Gamma$$
$$S_D(\lambda) = \varphi \mid_\Gamma = (p_S - \nu n \cdot \nabla u_S \cdot n) \mid_\Gamma$$

From these expressions discretizing with finite elements the Stokes velocity (u_S) with Galerkin approximation:

$$S_S(\lambda) = \sum_j \int_\Gamma (p_S - n \cdot \nu \nabla(\delta u_S)(u_S)_j \cdot n)$$
$$S_D(\lambda) = \sum_j \int_\Gamma (p_S - \nu n \cdot \nabla(\delta u_S)(u_S)_j \cdot n)$$

It can be seen that the Dirichlet-to-Neumann operators corresponds to the boundary inegral terms in the finite element discretization for both Stokes and Darcy's equations.

2. Write down the matrix form of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems.

   Note that in this coupled problem to different unknowns are, present, so the vector of unknowns will have the interior and boundary contributions of both. In a compact notation the matrix formulation will look like:

$$\begin{bmatrix} A_{II} & B_{I\Gamma} & A_{I\Gamma} & 0 & 0 \\ B_{\Gamma I} & 0 & B_{I\Gamma} & 0 & 0 \\ A_{\Gamma I} & B_{\Gamma I} & A_{\Gamma\Gamma} & M_{\Gamma\Gamma} & 0 \\ 0 & 0 & -M_{\Gamma\Gamma} & A_{\Gamma\Gamma} & A_{I\Gamma} \\ 0 & 0 & 0 & A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_\Omega \\ p \\ u_\Gamma \\ \varphi_\Gamma \\ \varphi_\Omega \end{bmatrix}$$

4 Monolithic and partitioned schemes in time

Consider the one-dimensional, transient, heat transfer equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \quad in[0,1]$$
$$u(x=0,t) = 0$$
$$u(x=1,t) = 0$$
$$u(x,t=0) = 0$$

1. Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\kappa = 1, f = 1, \delta t = 1$.

First of all lets find the weak form of the problem:

$$\int_0^1 v\frac{\partial u}{\partial t} + \int_0^1 k\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial x}|_0^1 = \int_0^1 vf$$

And replacing the space and time discretization:

$$u = \sum_j v_j u_j$$

$$\frac{\partial u}{\partial t} = \frac{u^{n+1} - u^n}{\delta t}$$

So the matrix for of the problem is, considering $\kappa = 1, f = 1, \delta t = 1$:

$$\sum_{ij}\int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij}\int_{\Omega^e}\frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} - \sum_{ij}\int_{\partial\Omega} v_i\frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i\int_{\Omega^e} v_i$$

2. Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements ($h = 0.2$), while the right subdomain is composed of 3 elements ($h = 0.2$). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as $u_0; u_1, u_2; u_3, u_4; u_5$. The interface is at u_2.

   The continuity restriction of u requires that through the interface:

$$u_{u_2} = 0$$

   On the other hand the second transmission condition reads:

$$\frac{\partial u_{\Gamma 12}}{\partial x} - \frac{\partial u_{\Gamma 21}}{\partial x} = 0$$

   Therefore on the interface ($x = 0.4$) the boundary integral will have contributions from both subdomains, and replacing the second transmission condition:

$$\frac{\partial v_j u_j^{n+1}}{\partial x} - \sum_{ij}\int_{\Gamma 12} v_i\frac{\partial v_j u_j^{n+1}}{\partial x} + \frac{\partial v_j u_j^{n+1}}{\partial x} - \sum_{ij}\int_{\Gamma 21} v_i\frac{\partial v_j u_j^{n+1}}{\partial x} = 0$$

   The boundary integrals are the same because the derivatives have to be the same and also the value of the unknown. So there is no need to integrate on the interface.

3. Obtain the algebraic form of the Dirichlet-to-Neumann operator (Steklov-Poincaré's operator) for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

The Dirichlet-to-Neumann operator in the problem corresponds to the gradint of temperatures on the interface. In the left subdomain the gradient of temperatures can be written in an approximate way using Taylor expansion series:

$$S(u) = \frac{\partial u_2^{n+1}}{\partial x} \approx \frac{u_2^{n+1} - u_1^{n+1}}{h} = \frac{u_2^{n+1} - (u_1^n + \frac{\partial u_1^n}{\partial t}\delta t)}{h}$$

4. Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = \kappa \partial x u^{n+1}$ at the coordinate of node 2.

   The Neumann-to-Dirichlet in the right hand side consists in determine the value of temperature u_2^{n+1} form the flux ϕ^{n+1} and values of temperatures at time step u^n.

$$S(\phi) = u_2^{n+1} \approx u_3^{n+1} - \phi^{n+1}h = (u_3^n + \frac{\partial u_3^n}{\partial t}\delta t) - \phi^{n+1}h$$

5. Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.

   The described iterative scheme with Dirichlet boundary conditions on the left interface and Neumann on the right one, implies that the temperatures in the left domain are imposed with the Neumann-to-Dirichlet operator, and the fluxes in the right domain are computed with the Dirichlet-to-Neumann operator in the left domain.

   In a staggered approach the predicted values are moved to the right hand side.

$Left:$

$$\sum_{ij} \int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij} \int_{\Omega^e} \frac{\partial v_i}{\partial x} \frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij} \int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x}$$

$${u_2^{n+1}}_{left} = N - t - D(right)$$

$$\sum_{ij} \int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij} \int_{\Omega^e} \frac{\partial v_i}{\partial x} \frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij} \int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x}$$

$${\frac{\partial u_2}{\partial x}}_{right} = D - t - N(right)$$

6. Do the same for a substitution and an iteration by subdomains scheme.

In the substitution scheme the already calculated fluxes at the interface values from the left are substituted as known values to the right domain, therefore there is no need of Dirichlet-to-Neumann prediction for the temperature at the interface in the right domain.

$$Left:$$
$$\sum_{ij}\int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij}\int_{\Omega^e} \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij}\int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x}$$
$$u_2^{n+1}{}_{left} = N - t - D(right)$$
$$Right:$$
$$\sum_{ij}\int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij}\int_{\Omega^e} \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij}\int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x}$$
$$\frac{\partial u_2^{n+1}}{\partial x}_{right} = \frac{\partial u_2^{n+1}}{\partial x}_{left}$$

On the other hand the iterative procedure is equivalent to the substituation one but iterating between subdomains till convergence is achieved.

$$Left:$$
$$\sum_{ij}\int_{\Omega^e} v_i v_j u_j^{n+1} + \sum_{ij}\int_{\Omega^e} \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij}\int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x} + \sum_{ij}\int_{\Omega^e} v_i v_j u_j^n$$
$$(u_2^{n+1}{}_{left})^i = (u_2^{n+1}{}_{right})^{i-1}$$
$$Right:$$
$$\sum_{ij}\int_{\Omega^e} v_i v_j u_j^{n+1} + \sum_{ij}\int_{\Omega^e} \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij}\int_{\partial\Omega} v_i \frac{\partial v_j \tilde{u}_j^{n+1}}{\partial x} + \sum_{ij}\int_{\Omega^e} v_i v_j u_j^n$$
$$(\frac{\partial u_2}{\partial x}_{right})^i = (\frac{\partial u_2}{\partial x}_{left})^i$$

7. Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface), using Nitsche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

   The Nitsche's method considers a penalty parameter α that multiplies two additional integrals that assures the imposition of Dirichlet boundary condition in the interface:

$$\alpha \sum_{ij}\int_\Gamma v_i v_j u_j - \frac{\alpha\kappa}{h} - \sum_{ij}\int_\Gamma \frac{\partial v_i}{\partial x} v_j u_j$$

In this way the algebraic system to solve has the form:

$$\sum_{ij}\int_{\Omega^e} v_i(v_j u_j^{n+1} - v_j u_j^n) + \sum_{ij}\int_{\Omega^e} \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} - \sum_{ij}\int_\Gamma v_i \frac{\partial v_j u_j^{n+1}}{\partial x} +$$

$$\alpha \sum_{ij}\int_\Gamma v_i v_j u_j - \frac{\alpha\kappa}{h}\sum_{ij}\int_\Gamma \frac{\partial v_i}{\partial x} v_j u_j = \sum_i \int_{\Omega^e} v_i + \alpha \sum_{ij}\int_\Gamma v_i v_j \bar{u}_j - \frac{\alpha\kappa}{h}\sum_{ij}\int_\Gamma \frac{\partial v_i}{\partial x} v_j \bar{u}_j$$

5 Operator splitting techniques

Consider the one dimensional, transient, convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{0lx^2} + a_x \frac{\partial u}{\partial x} = f \quad in[0,1]$$
$$u(x=0,t) = 0$$
$$u(x=1,t) = 0$$
$$u(x,t=0) = 0$$

with $\kappa = 0, a_x = 1, f = 1$.

1. Discretize it in space using finite elements (3 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size δt.

   The weak form of the previous problem is:

   $$\int_0^1 v\frac{\partial u}{\partial t} + \int_0^1 k\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial x}|_0^1 - a_x \int_0^1 v\frac{\partial u}{\partial x} = \int_0^1 vf$$

   And the discretization with finite elements and BDF1 in time:

   $$\int_0^1 v_i \frac{v_j(u_j^{n+1} - u_j^n)}{\delta t} + \int_0^1 k\frac{\partial v_i}{\partial x}\frac{\partial v_j u_j}{\partial x} - v_i\frac{\partial v_j u_j}{\partial x}|_0^1 - a_x \int_0^1 v_i \frac{\partial v_j u_j}{\partial x} = \int_0^1 v_i f$$

   which results in the system:

   $$\sum_{ij}\int_{\Omega^e}\left(\frac{v_i v_j}{\delta t} + \kappa\frac{\partial v_i}{\partial x}\frac{\partial v_j}{\partial x} - a_x v_i \frac{\partial v_j}{\partial x}\right) u_j^{n+1} = \sum_i \int_{\Omega^e} v_i f + \sum_{ij}\int_{\Omega^e}\left(\frac{v_i v_j}{\delta t}\right) u_j^n$$

   The integrals in the element for the system matrix are the followings:

   $$K_{11}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_2^e x + (x_2^e)^2\right) + \kappa x + a_x x\left(x_2^e - \frac{x}{2}\right)\right]_{x_1}^{x_2}$$

   $$K_{22}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_1^e x + (x_1^e)^2\right) + \kappa x + a_x x\left(\frac{x}{2} - x_1^e\right)\right]_{x_1}^{x_2}$$

   $$K_{12}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1 x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right) - \kappa x + a_x x\left(\frac{x}{2} - x_1\right)\right]_{x_1}^{x_2}$$

   $$K_{21}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1 x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right) - \kappa x - a_x x\left(\frac{x}{2} - x_2\right)\right]_{x_1}^{x_2}$$

And the right hand side:

$$f_1^e = \left[fx\left(x_2 - \frac{x}{2}\right)\right]_{x_1}^{x_2} + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_2^e x + (x_2^e)^2\right)\right] u_1^n + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1 x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right)\right] u_2^n$$

$$f_2^e = \left[fx\left(\frac{x}{2} - x_1\right)\right]_{x_1}^{x_2} + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1 x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right)\right] u_1^n + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_1^e x + (x_1^e)^2\right)\right] u_2^n$$

Then the element system equation is:

$$\begin{bmatrix} \frac{1}{9\delta t} + \frac{7}{2} & \frac{-1}{18\delta t} - \frac{5}{2} \\ \frac{-1}{18\delta t} - \frac{5}{2} & \frac{1}{9\delta t} + \frac{7}{2} \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{f}{18} \\ \frac{f}{18} \end{bmatrix} + \begin{bmatrix} \frac{1}{9\delta t} & -\frac{1}{18\delta t} \\ -\frac{1}{18\delta t} & \frac{1}{9\delta t} \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \end{bmatrix}$$

Assembling the three element matrices and replacing the boundary conditions ($u_0 = u_3 = 0$) and initial conditions ($u(x, t = 0) = u^n = 0$) the resulting system is:

$$\begin{bmatrix} \frac{1}{9\delta t} + \frac{7}{2} & \frac{-1}{18\delta t} - \frac{5}{2} & 0 & 0 \\ \frac{-1}{18\delta t} - \frac{5}{2} & \frac{2}{9\delta t} + 7 & \frac{-1}{18\delta t} - \frac{5}{2} & 0 \\ 0 & \frac{-1}{18\delta t} - \frac{5}{2} & \frac{2}{9\delta t} + 7 & \frac{-1}{18\delta t} - \frac{5}{2} \\ 0 & 0 & \frac{-1}{18\delta t} - \frac{5}{2} & \frac{1}{9\delta t} + \frac{7}{2} \end{bmatrix} \begin{bmatrix} 0 \\ u_1^{n+1} \\ u_2^{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{9} \\ \frac{1}{9} \\ \frac{1}{18} \end{bmatrix}$$

And the reduced system:

$$\begin{bmatrix} \frac{2}{9\delta t} + 7 & \frac{-1}{18\delta t} - \frac{5}{2} \\ \frac{-1}{18\delta t} - \frac{5}{2} & \frac{2}{9\delta t} + 7 \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \end{bmatrix}$$

Clearly the solutions have to be the same in the nodes 1 and 2 since the equations are the same:

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{2\delta t}{81\delta t + 3} \\ \frac{2\delta t}{81\delta t + 3} \end{bmatrix}$$

2. Solve the same time step by using a first order operator splitting technique.

   To use an operator splitting first we have to split our problem in transient, convective and diffusion matrices, as:

$$K_{transient} = \sum_{ij} \int_{\Omega^e} \left(\frac{v_i v_j}{\delta t}\right)$$

$$K_{diffusion} = \sum_{ij} \int_{\Omega^e} \kappa \frac{\partial v_i}{\partial x} \frac{\partial v_j}{\partial x}$$

$$K_{convection} = -\sum_{ij} a_x v_i \frac{\partial v_j}{\partial x}$$

   And the splitting approach to the solution of the problem is first solve the convective term with zero force vector and initial condition in the previous time step and after that solve the diffusion term with the force vector and initial condition the convective velocity already computed:

$$K_{transient}u_c + K_{convection}u_c = 0$$
$$K_{transient}u_d + K_{diffusion}u_d = f$$

The algebraic setting of the first time step of the convective problem is:

$$\begin{bmatrix} \frac{2}{9\delta t}+1 & \frac{-1}{18\delta t}+\frac{1}{2} \\ \frac{-1}{18\delta t}+\frac{1}{2} & \frac{2}{9\delta t}+1 \end{bmatrix} \begin{bmatrix} (u_1^{n+1})_c \\ (u_2^{n+1} =_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which once solved is:

$$\begin{bmatrix} (u_1^{n+1})_c \\ (u_2^{n+1})_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now solving the diffusion problem with the initial condition of the convection temperature at time step $n+1$, this leaves:

$$\begin{bmatrix} \frac{2}{9\delta t}+6 & \frac{-1}{18\delta t}-3 \\ \frac{-1}{18\delta t}-3 & \frac{2}{9\delta t}+6 \end{bmatrix} \begin{bmatrix} (u_1^{n+1})_d \\ (u_2^{n+1})_d \end{bmatrix} = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \end{bmatrix}$$

And once solved, the final solution is obtained:

$$\begin{bmatrix} (u_1^{n+1})_c \\ (u_2^{n+1})_c \end{bmatrix} = \begin{bmatrix} \frac{2\delta t}{54\delta t+1} \\ \frac{2\delta t}{54\delta t+1} \end{bmatrix}$$

which as seen is not the same solution obtained with the monolithic approach.

3. Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $t = 1, t = 0.5, t = 0.25$. Comment on the results.

   As seen in Figure 1 the solution doesn't converge because as the time step decreases the error becomes higher and higher. On the other hand it is observed that the error tends to stabilise for large time steps. This behaviour can be understood because the solution we obtained for both methods is divided by a δt that when decreases makes the quotient grow, that is why for small time stepping the error increases.

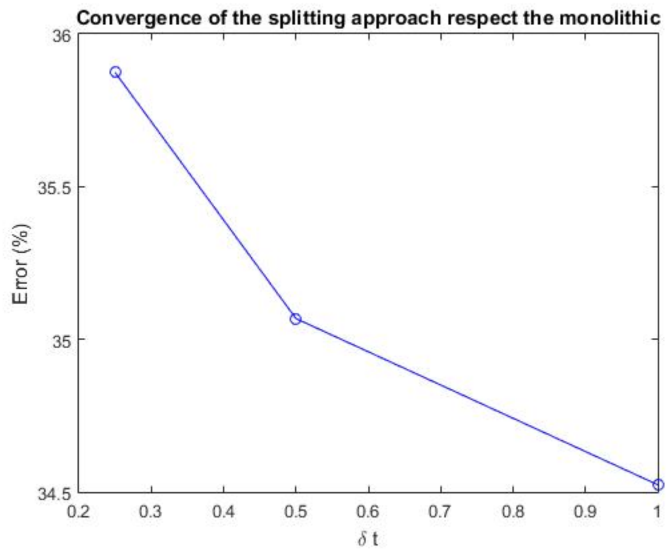

Figure 1: Convergence of the error of the splitting approach respect the monolithic solution.

6 Fractional step method

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$M\frac{1}{\delta t}\left(\hat{U}^{n+1}-U^{n}\right)+K\hat{U}^{n+1}=f-G\tilde{P}^{n+1}$$
$$DM^{-1}GP^{n+1}=\frac{1}{\delta t}D\hat{U}^{n+1}-DM^{-1}G\hat{P}^{n-1}$$
$$M\frac{1}{\delta t}\left(U^{n+1}-\hat{U}^{n+1}\right)+\alpha K\left(U^{n+1}-\hat{U}^{n+1}\right)+G\left(P^{n+1}-\tilde{P}^{n-1}\right)=0$$

1. Which is the optimal value for the α parameter?

   If $\alpha=\frac{2}{3}$ the Yosida time discretization corresponds to BDF2 which requires the solution at $n+1$ and n, that is the information we have. The BDF2 method is third-order in time and is the best approximation we can have with the method proposed.

2. What is the source of error of the scheme?

   The Yosida scheme uses an inexact LU factorisation of the matrix arrising from the discretization with finite elements in space and with finite differences in time. In this way the system is split and it can be computed in a fractional step setting. The problem is that, since the factorisation is inexact and also predictions on the pressure at time step $n+1$ for solving the velocity, the method introduces errors in the final solution.

7 ALE formulation

7.1 Problem 1

Given the spatial description of a property

$$\gamma(x, y, z, t) = [2x, ye^t, z]$$

the equations of movement:

$$\begin{aligned} x &= Xe^t \\ y &= Y + e^t - 1 \\ z &= Z \end{aligned}$$

and the equations of the movement of the mesh:

$$\begin{aligned} x_m &= \mathcal{X} + \alpha t \\ y_m &= \mathcal{Y} - \beta t \\ z_m &= \mathcal{Z} \end{aligned}$$

1. Obtain the description of the property in terms of the ALE coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

   To obtain the ALE description of the property γ we start from the initial reference configuration, described by the material coordinates X. At the initial reference configuration the ALE coordinates $\mathcal{X}$ coincide with the material coordinates (before displacement of both mesh and particles). Therefore first of all we express the property γ in material coordinates using the equation of movenent:

   $$\gamma(X, Y, Z, t) = [2(Xe^t), (Y + e^t - 1)e^t, Z] = [2Xe^t, Ye^t + e^{2t} - e^t, Z]$$

   Now to obtain the ALE description of the property we must substitute in the material coordinates the description of the movement of the mesh to obtain the ALE description:

   $$\gamma(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, t) = [2(\mathcal{X} + \alpha t)e^t, (\mathcal{Y} - \beta t)e^t + e^{2t} - e^t, \mathcal{Z}]$$

2. Compute the velocity of the particles and the mesh velocity.

The velocity of the particles is given by the time derivative of the equation of movement, which in material coordinates is:

$$\frac{\partial x(X)}{\partial t} = Xe^t$$
$$\frac{\partial y(X)}{\partial t} = e^t$$
$$\frac{\partial z(X)}{\partial t} = 0$$

On the other hand the mesh velocity is given in ALE description as the time derivative of the mesh movement:

$$\frac{\partial x_m}{\partial t} = \alpha$$
$$\frac{\partial y_m}{\partial t} = -\beta$$
$$\frac{\partial z_m}{\partial t} = 0$$

7.2 Problem 2

Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?

The incompressible Navier-Stokes equation in ALE description is obtained evaluating the velocities and pressure instead of in the material description in the ALE reference $(u_{\mathcal{X}}, p_{\mathcal{X}})$. The time derivative is computed with the total time derivative that accounts for the movement of the mesh with velocity v_{mesh} and the movemenet of the domain with velocity (v). Therefore the ALE description of the incompressible Navier-Stokes equation is:

$$\frac{\partial u_{\mathcal{X}}}{\partial t} + [(v - v_{mesh}) \cdot \nabla]u_{\mathcal{X}} - \nu \Delta p_{\mathcal{X}} u_{\mathcal{X}} + \nabla p_{\mathcal{X}} = f_{\mathcal{X}}$$
$$\nabla \cdot u_{\mathcal{X}} = 0$$

All velocities and pressure are evaluated in the ALE domain excepts for the mesh velocity which is evaluated with respect to the reference configuration. Here the time derivative formula for the ALE configuration as been used:

$$\frac{du_{\mathcal{X}}}{dt} = \frac{\partial u_{\mathcal{X}}}{\partial t} + (v - v_{mesh}) \cdot \nabla u_{\mathcal{X}}$$

7.3 Problem 3

Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods.

The ALE formulation is widely used in problems where the displacements or deformations are important and also the way how boundary conditions are applied. In a moving domain the orientation and shape of the boundary may change and in coupled problems where transmission conditions have to be imposed between subdomains, it is important to assure that they are applied at the right boundary each moment in time.

This is the case for instance of fluid-structure interactions where the imposition of interface conditions depend on the position in a bidirectional way. Therefore it is not possible to impose these conditions assuming a rigid body and ALE is well suitable to this end.

Another example where ALE description is used is the free surface computation in fluid coupling. When only the normal component of the velocity in the boundary is imposed the new position of the mesh is computed with the normal component of the velocity and the remeshing is done in the tangent of it.

Other uses in many physical problems where a two side interaction is done are found for ALE formulation. Contact problems of deformable solids also need of an ALE description to be able to compute the deformation and stress states in both domains in each moment. Also in nonlinear solid mechanics ALE is also necessary to capture fast and dramatic changes in the shape of the solid. If ALE is not used the elements can become too distorted to produce good and accurate solutions. In this case an ALE remeshing strategy is useful to avoid distortion and capture the real phenomena under study.

8 Fluid-structure interaction

8.1 Problem 1

Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

The added mass effect is experimented when a coupled problem solved iteratively doesn't converge. This effect can occur when solving the coupling of incompressible fluids with solids, specially when the density of both is not very different. This can be the case for instance in biomedics of blood-tissue interaction, where the light density of the tissue is comparable to the one of blood.

The term added mass is introduced because it is derived from the lumped mass matrix used to solve the coupling of two domains. In this case when the value of one of the unknowns is imposed as a boundary condition, the known quantity can be moved to the right hand side vector in an additional mass term, naming the so called added mass operator.

From a stability analysis it is derived that the convergence of the iterative procedure depends on the relation between densities of the solid and the fluid, which has to satisfies an inequality. If it is not the case the algorithm may not converge to the solution and be unstable.