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CP HW 1

Transmission Conditions

1.) Deflection $u(x)$ of Euler-Bernoulli beam, governed by

$$EI \frac{d^4 u}{dx^4} = f$$

$EI \rightarrow$ Mechanical property of section and material

$f \rightarrow$ Distributed load

Beam is clamped at $x = 0$, $x = L$

$$\text{PVW} \longrightarrow EI \int_0^L \frac{d^2 \delta u}{dx^2} \frac{d^2 u}{dx^2} = \int_0^L \delta u \, f$$

for all δu such that $\delta u(0) = \delta u(L) = 0$, $\frac{d\delta u}{dx}(0) = \frac{d\delta u}{dx}(L) = 0$

a.) Postulate the space of functions where both u and δu must belong

For the problem to make sense, we must guarantee that the integrals of the weak form are bounded.

$$\int_0^L \frac{d^2 \delta u}{dx^2} \frac{d^2 u}{dx^2} < \infty \quad \text{(Bounded)}$$

Therefore, u and δu must exist in the space where their second derivatives are square integrable

$$\boxed{u, \delta u \in H^2(\Omega)} \qquad H^2(\Omega) = \left\{ u \in L^2(\Omega) : \frac{du}{dx} \in L^2(\Omega) : \frac{d^2 u}{dx^2} \in L^2(\Omega) \right\}$$

b.) If $[0, L] = [0, P] \cup [P, L]$, obtain the transmission conditions at P implied by regularity requirements.

$u \in H^2(\Omega) \longrightarrow u$ and $\frac{du}{dx}$ are continuous in Ω

$$\left.\begin{aligned} [\![u]\!] = 0 \quad & \text{1st transmission condition} \\ \left[\!\left[\frac{du}{dx}\right]\!\right] = 0 \quad & \text{2nd transmission condition} \end{aligned}\right\} \text{at } P$$

c) Obtain the transmission conditions at P that follow by imposing in the PVW that the integral is additive

$$EI\int_0^L \frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2} = \int_0^L \delta u f\, dx$$

Imposing additivity $\longrightarrow EI\int_0^P \frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2}dx + EI\int_P^L \frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2}dx = \int_0^L \delta u f\, dx$

- Integrating the 1st term by parts: (Domain 1)

$$EI\left[\left[\frac{d\delta u}{dx}\frac{d^2u}{dx^2}\right]_0^P - \int_0^P \frac{d\delta u}{dx}\frac{d^3u}{dx^3}dx\right] = EI\left[\left[\frac{d\delta u}{dx}\frac{d^2u}{dx^2}\right]_0^P - \left[\delta u\frac{d^3u}{dx^3}\right]_0^P + \int_0^P \delta u\frac{d^4u}{dx^4}dx\right]$$

$$= EI\left[\frac{d\delta u}{dx}\frac{d^2u}{dx^2}\right]_0^P - EI\left[\delta u\frac{d^3u}{dx^3}\right]_0^P + \int_0^P \delta u\underbrace{\left(EI\frac{d^4u}{dx^4}\right)}_{f}dx$$

- Domain 2:

$$EI\left[\frac{d\delta u}{dx}\frac{d^2u}{dx^2}\right]_P^L - EI\left[\delta u\frac{d^3u}{dx^3}\right]_P^L + \int_P^L \delta u\left(EI\frac{d^4u}{dx^4}\right)dx$$

$\delta u(0) = \delta u(L) = 0$

$\frac{d\delta u}{dx}(0) = \frac{d\delta u}{dx}(L) = 0$

Adding both:

$$EI\left[\frac{d\delta u_1}{dx}\frac{d^2u_1}{dx^2}\right]_P - EI\left[\delta u_1\frac{d^3u_1}{dx^3}\right]_P + EI\left[\frac{d\delta u_2}{dx}\frac{d^2u_2}{dx^2}\right]_P - EI\left[\delta u_2\frac{d^3u_2}{dx^3}\right]_P$$

$$+ \cancel{\int_0^P \delta u f\, dx} + \cancel{\int_P^L \delta u f\, dx} = \cancel{\int_0^L \delta u f\, dx}$$

$$EI\left[\frac{d\delta u_1}{dx}\frac{d^2u_1}{dx^2}\right]_P - EI\left[\delta u_1\frac{d^3u_1}{dx^3}\right] + EI\left[\frac{d\delta u_2}{dx}\frac{d^2u_2}{dx^2}\right]_P - EI\left[\delta u_2\frac{d^3u_2}{dx^3}\right]_P = 0$$

Since δu is arbitrary:

$$[\![EI\frac{d^3u_1}{dx^3}]\!] = 0 \qquad [\![EI\frac{d^2u}{dx^2}]\!] = 0 \longrightarrow \text{weak continuity}$$

2) Maxwell problem $u: \Omega \longrightarrow \mathbb{R}^3$

$$\nu \nabla \times \nabla \times u = f \quad \text{in } \Omega$$
$$\nabla \cdot u = 0 \quad \text{in } \Omega$$
$$n \times u = 0 \quad \text{on } \partial\Omega$$

$\nu > 0$

f is a divergence free force field

n is the unit exterior normal

a) Write a variational statement.

$$\int_\Omega \delta u \cdot (\nu \underbrace{\nabla \times \nabla \times u}_{\omega}) = \int_\Omega \delta u \, f$$

$$\int_\Omega \delta u \cdot (\nu \nabla \times \omega) = \int_\Omega \delta u \, f \qquad (1)$$

considering that

$$u \cdot (\nabla \times v) = v \cdot \nabla \times u - \nabla \cdot (u \times v)$$

(1) takes the form:

$$\int_\Omega (\nu\omega) \cdot (\nabla \times \delta u) - \int_\Omega \nabla \cdot (\delta u \times \nu\omega) = \int_\Omega \delta u \, f$$

Applying divergence theorem:

$$\nu \int_\Omega (\nabla \times \delta u) \cdot \omega - \int_{\partial\Omega} (\delta u \times \nu\omega) \cdot n \, d\Gamma = \int_\Omega \delta u \, f \qquad (2)$$

Recalling that

$$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)$$

(2) becomes:

$$\nu \int_\Omega (\nabla \times \delta u) \cdot \omega - \nu \int_{\partial\Omega} \omega \cdot (n \times \delta u) = \int_\Omega \delta u \, f$$

$$\boxed{\nu \int_\Omega (\nabla \times \delta u) : (\nabla \times u) = \int_\Omega \delta u \, f + \nu \int_{\partial\Omega} \omega \cdot (n \times \delta u)}$$

For the integral to be bounded, the curl of u and δu must be square integrable, therefore:

$$u, \delta u \in H(curl, \Omega)$$

$$H(curl, \Omega) = \{ u \in L^2(\Omega) : \nabla \times u \in L^2(\Omega) \}$$

b.) $\Gamma \rightarrow$ surface that intersects Ω. obtain transmission conditions across Γ implied by regularity requirements

$$\int_{x_0-a}^{x_0+a} (\nabla \times u)^2 = \lim_{\epsilon \to 0} \int_{x_0-a}^{x_0+a} (\nabla \times u^\epsilon)^2$$

$$\int_{x_0-a}^{x_0+a} (\nabla \times u^\epsilon)^2 = \int_{x_0-a}^{x_0-\frac{\epsilon}{2}} (\nabla \times u)^2 + \int_{x_0+\epsilon/2}^{x_0+a} (\nabla \times u)^2 + \epsilon \left[\frac{n \times u_{(x_0+\epsilon/2)} - n \times u_{(x_0-\epsilon/2)}}{\epsilon} \right]^2$$

$$\lim_{\epsilon \to 0} \int_{x_0-a}^{x_0+a} (\nabla \times u)^2 = \infty$$

Therefore, for the integral to be bounded $\longrightarrow \boxed{[\![n \times u]\!] = 0}$

The tangential components of u must be continuous across Γ.

c.) Obtain the transmission conditions across Γ imposing that the integral is additive.

$$\nu \int_\Omega (\nabla \times \delta u) \cdot (\nabla \times u) - \nu \int_{\partial\Omega} (\nabla \times u) \cdot (n \times \delta u) = \int_\Omega \delta u f \qquad (3)$$

- at Ω_1:

$$\nu \int_{\Omega_1} (\nabla \times \delta u) \cdot (\nabla \times u) - \nu \int_{\partial\Omega_1} (\nabla \times u) \cdot (n \times \delta u) - \nu \int_\Gamma (\nabla \times u_1) \cdot (n_1 \times \delta u) = \int_{\Omega_1} \delta u f$$

- at Ω_2

$$\nu \int_{\Omega_2} (\nabla \times \delta u) \cdot (\nabla \times u) - \nu \int_{\partial\Omega_2} (\nabla \times u) \cdot (n \times \delta u) - \nu \int_\Gamma (\nabla \times u_2) \cdot (n_2 \times \delta u) = \int_{\Omega_2} \delta u f$$

Adding both equations and comparing to ③ yields:

$$\nu \int_\Gamma \delta u \cdot (\nabla \times u_1 \times n_1) + \delta u \cdot (\nabla \times u_2 \times n_2) = 0$$

$$\boxed{[\![\nu (\nabla \times u \times n)]\!]_\Gamma = 0}$$

3.) Navier equations for an elastic material can be written as:

$$-2\mu \nabla \cdot (\varepsilon(u)) - \lambda \nabla(\nabla \cdot u) = \rho b$$

$$-\mu \Delta u - (\lambda + \mu) \nabla(\nabla \cdot u) = \rho b \qquad u = 0 \text{ on } \partial\Omega$$

$$\mu \nabla \times (\nabla \times u) - (\lambda + 2\mu) \nabla(\nabla \cdot u) = \rho b$$

$u \longrightarrow$ Displacement field $\qquad \varepsilon(u) = \nabla^s u \qquad \lambda, \mu \to$ Lamé coef.

$\rho \to$ Density $\qquad b \longrightarrow$ Body forces

a) Variational form in the appropriate functional spaces

- 1st equation

$$-2\mu \int_\Omega \delta u \, \nabla \cdot (\varepsilon) - \lambda \int_\Omega \delta u \, \nabla(\nabla \cdot u) = \int_\Omega \delta u \, \rho b$$

$$-2\mu \left[\int_{\partial\Omega} n \cdot (\varepsilon \, \delta u) - \int_\Omega \nabla \delta u : \varepsilon \right] - \lambda \left[\int_{\partial\Omega} n \cdot (\delta u \, (\nabla \cdot u)) - \int_\Omega (\nabla \cdot \delta u)(\nabla \cdot u) \right] = \int_\Omega \delta u \cdot \rho b$$

$$\boxed{2\mu \int_\Omega (\nabla \delta u) : \varepsilon + \lambda \int_\Omega (\nabla \cdot \delta u)(\nabla \cdot u) - 2\mu \int_{\partial\Omega} n \cdot (\varepsilon \, \delta u) - \lambda \int_{\partial\Omega} n \cdot (\delta u (\nabla \cdot u)) = \int_\Omega \delta u \cdot \rho b}$$

since $\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \longrightarrow \nabla u$ is the most restrictive function in the expression.

For the integral to be in $L^2(\Omega) \longrightarrow u, \delta u \in H^1(\Omega)$

• 2nd equation

$$-\mu\int_{\Omega}\delta u\cdot\Delta u-(\lambda+\mu)\int_{\Omega}\delta u\,\nabla(\nabla\cdot u)=\int_{\Omega}\delta u\cdot\rho b$$

$$-\mu\left[\int_{\partial\Omega}n\cdot(\delta u\cdot\nabla u)-\int_{\Omega}\nabla\delta u:\nabla u\right]-(\lambda+\mu)\left[\int_{\partial\Omega}n\cdot(\delta u(\nabla\cdot u))-\int_{\Omega}(\nabla\cdot\delta u)(\nabla\cdot u)\right]=\int_{\Omega}\delta u\cdot\rho b$$

$$\boxed{\mu\int_{\Omega}\nabla\delta u:\nabla u+(\lambda+\mu)\int_{\Omega}(\nabla\cdot\delta u)(\nabla\cdot u)-\mu\int_{\partial\Omega}n\cdot(\delta u\cdot\nabla u)-(\lambda+\mu)\int_{\partial\Omega}n\cdot(\delta u(\nabla\cdot u))=\int_{\Omega}\delta u\cdot\rho b}$$

$$\nabla\delta u,\ \nabla u\in L^2(\Omega)\longrightarrow u,\delta u\in H^1(\Omega)$$

• 3rd equation:

$$\mu\int_{\Omega}\delta u\cdot(\nabla\times\underbrace{(\nabla\times u)}_{\omega})-(\lambda+2\mu)\int_{\Omega}\delta u\cdot\nabla(\nabla\cdot u)=\int_{\Omega}\delta u\cdot\rho b$$

$$\mu\int_{\Omega}\delta u\cdot(\nabla\times\omega)-(\lambda+2\mu)\left[\int_{\partial\Omega}n\cdot(\delta u(\nabla\cdot u))-\int_{\Omega}(\nabla\cdot\delta u)(\nabla\cdot u)\right]=\int_{\Omega}\delta u\cdot\rho b$$

$$\mu\left[\int_{\Omega}\omega\cdot(\nabla\times\delta u)-\int_{\Omega}\nabla\cdot(\delta u\times\omega)\right]-(\lambda+2\mu)\left[\int_{\partial\Omega}n\cdot(\delta u(\nabla\cdot u))-\int_{\Omega}(\nabla\cdot\delta u)(\nabla\cdot u)\right]=\int_{\Omega}\delta u\cdot\rho b$$

$$\boxed{=\mu\int_{\Omega}(\nabla\times u)\cdot(\nabla\times\delta u)+(\lambda-2\mu)\int_{\Omega}(\nabla\cdot\delta u)(\nabla\cdot u)-\mu\int_{\partial\Omega}\omega\cdot(n\times\delta u)-(\lambda-2\mu)\int_{\partial\Omega}n\cdot(\delta u(\nabla\cdot u))=\int_{\Omega}\delta u\cdot\rho b}$$

$(\nabla\times u),(\nabla\cdot u)\in L^2(\Omega)\longrightarrow$

$u,\delta u\in H(\text{curl},\Omega)\cap H(\text{div},\Omega)$

$(\nabla\times\delta u),(\nabla\cdot\delta u)\in L^2(\Omega)\longrightarrow$

$$H(\text{curl},\Omega)=\{u\in L^2(\Omega):\nabla\times u\in L^2(\Omega)\}$$

$$H(\text{div},\Omega)=\{u\in L^2(\Omega):\nabla\cdot u\in L^2(\Omega)\}$$

b.) Obtain the transmission conditions across Γ that follow by imposing that the integral is additive

• 1st eq.

• at Ω_1 　　　　imposing $\delta u=0$ at $\partial\Omega$

$$2\mu\int_{\Omega_1}(\nabla\delta u):\varepsilon+\lambda\int_{\Omega_1}(\nabla\cdot\delta u)(\nabla\cdot u)-\cancel{2\mu\int_{\partial\Omega_1}n\cdot(\varepsilon\,\delta u)}-2\mu\int_{\Gamma}n_1\cdot(\varepsilon\,\delta u)$$

$$-\lambda\int_{\partial\Omega_1}n\cdot(\delta u(\nabla\cdot u))-\lambda\int_{\Gamma}n_1\cdot(\delta u(\nabla\cdot u_1))=\int_{\Omega_1}\delta u\,\rho b$$

- at Ω_2 following the same process

$$2\mu\int_{\Omega_2}(\nabla\delta u):\varepsilon + \lambda\int_{\Omega_2}(\nabla\cdot\delta u)(\nabla\cdot u) - 2\mu\int_{\Gamma} n_2\cdot(\varepsilon\,\delta u) - \lambda\int_{\Gamma} n_2\cdot(\delta u(\nabla\cdot u_2)) = \int_{\Omega_2}\delta u\,\rho b$$

Adding both equations and comparing to the original weak form:

$$\underbrace{2\mu\int_{\Gamma} n_1\cdot(\varepsilon\,\delta u) + n_2\cdot(\varepsilon\,\delta u)}_{[\![n\cdot(\mu\varepsilon)]\!]=0} + \underbrace{\lambda\int_{\Gamma} n_1\cdot(\delta u(\nabla\cdot u_1)) + n_2\cdot(\delta u(\nabla\cdot u_2))}_{[\![n\cdot(\lambda\nabla\cdot u)]\!]=0} = 0$$

$$[\![n\cdot(\mu\varepsilon)]\!] = 0 \qquad [\![n\cdot(\lambda\nabla\cdot u)]\!] = 0$$

- 2nd equation:
  - at Ω_1

$$\mu\int_{\Omega_1}\nabla\delta u:\nabla u + (\lambda+\mu)\int_{\Omega_1}(\nabla\cdot\delta u)(\nabla\cdot u) - \mu\int_{\Gamma} n_1\cdot(\delta u\cdot\nabla u_1) - (\lambda+\mu)\int_{\Gamma} n_1\cdot(\delta u(\nabla\cdot u)) = \int_{\Omega_1}\delta u\,\rho b$$

- at Ω_2:

$$\mu\int_{\Omega_2}\nabla\delta u:\nabla u + (\lambda+\mu)\int_{\Omega_2}(\nabla\cdot\delta u)(\nabla\cdot u) - \mu\int_{\Gamma} n_2\cdot(\delta u\cdot\nabla u_2) - (\lambda+\mu)\int_{\Gamma} n_2\cdot(\delta u(\nabla\cdot u)) = \int_{\Omega_2}\delta u\,\rho b$$

Adding both equations yields:

$$\underbrace{\mu\int_{\Gamma}(n_1\cdot\nabla u_1 + n_2\cdot\nabla u)\,\delta u}_{[\![n\cdot(\mu\nabla u)]\!]=0} + \underbrace{(\lambda+\mu)\int_{\Gamma}(n_1(\nabla\cdot u_1) + n_2(\nabla\cdot u_2))\,\delta u}_{[\![(\lambda+\mu)(\nabla\cdot u)n]\!]=0} = 0$$

$$[\![n\cdot(\mu\nabla u)]\!] = 0 \qquad [\![(\lambda+\mu)(\nabla\cdot u)n]\!] = 0$$

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HW 2

Domain Decomposition Methods

1.) Euler-Bernoulli Beam $\longrightarrow EI\dfrac{d^4u}{dx^4} = f \quad \begin{cases} u(0) = u(L) = 0 \\ \dfrac{du(0)}{dx} = \dfrac{du(L)}{dx} = 0 \end{cases}$

$[0,L] = [0,L_1] \cup [L_2,L]$ with $L_2 < L_1$

a.) Iteration by subdomain scheme based on Schwarz additive domain decomp. method

$\Omega_{12} = \Omega_1 \cap \Omega_2 = [L_2, L_1]$ $\quad \Omega_1 = [0,L_1]$ $\quad \Omega_2 = [L_2, L]$ $\quad \partial\Omega_1 = 0$

$T_{12} = L_1$ $\quad T_{21} = L_2$ $\quad \partial\Omega_2 = L$

$$EI\frac{d^4u_1^k}{dx^4} = f \quad \text{in } \Omega_1$$

$$u_1^k = 0 \quad \text{on } \partial\Omega_1$$

$$\frac{du_1^k}{dx} = 0 \quad \text{on } \partial\Omega_1$$

$$u_1^k = u_2^{k-1} \quad \text{on } T_{12}$$

$$\frac{du_1^k}{dx} = \frac{du_2^{k-1}}{dx} \quad \text{on } T_{12}$$

$$EI\frac{d^4u_2^k}{dx^4} = f \quad \text{in } \Omega_2$$

$$u_2^k = 0 \quad \text{on } \partial\Omega_2$$

$$\frac{du_2^k}{dx} = 0 \quad \text{on } \partial\Omega_2$$

$$u_2^k = u_1^l \quad \text{on } T_{12}$$

$$\frac{du_2^k}{dx} = \frac{du_1^l}{dx} \quad \text{on } T_{12}$$

$$l = \begin{cases} k-1 & \text{for Jacobi (parallel) scheme} \\ k & \text{for Gauss-Seidel (sequential)} \end{cases}$$

b.) Matrix version of the previous scheme after FE discretization

Let us recall the weak form of the problem:

$$EI\int_0^L \frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2} = \int_0^L \delta u\, f$$

Replacing $\delta u = N$ and $u \approx u_i N_i$

$$EI\int_0^L \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}\, u_j = \int_0^L N_j f$$

Defining $A_{11} = a(N_i, N_j)$

$$Au = f$$

• at Ω_1:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_1^k \\ u_2^{k-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma \end{bmatrix}$$

• at Ω_2

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_2^k \\ u_1^l \end{bmatrix} = \begin{bmatrix} f_2 \\ f_\Gamma \end{bmatrix}$$

2) Maxwell problem. Let Γ be a surface that intersects Ω

$$\nu \nabla \times \nabla \times u = f \quad \text{in } \Omega$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \qquad u : \Omega \longrightarrow \mathbb{R}^3$$

$$n \times u = 0 \quad \text{on } \partial\Omega$$

a.) Iteration by subdomain based on Dirichlet - Neumann coupling

$$\begin{aligned} \nu_1 \nabla \times \nabla \times u_1^k &= f \quad \text{in } \Omega_1 \\ \nabla \cdot u_1^k &= 0 \quad \text{in } \Omega_1 \\ n \times u_1^k &= 0 \quad \text{on } \partial\Omega_1 \\ \nu_1(\nabla \times u_1^k \times n) &= \nu_2(\nabla \times u_2^{k-1} \times n) \quad \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} \nu_2 \nabla \times \nabla \times u_2^k &= f \quad \text{in } \Omega_2 \\ \nabla \cdot u_2^k &= 0 \quad \text{in } \Omega_2 \\ n \times u_2^k &= 0 \quad \text{on } \partial\Omega_2 \\ n \times u_2^k &= n \times u_1^l \quad \text{on } \Gamma \end{aligned}$$

$$l = \begin{cases} k-1 & \text{(Jacobi)} \\ k & \text{(Gauss-Seidel)} \end{cases}$$

b.) Steklov - Poincaré operator of the problem

Let us define u_i as $\longrightarrow$ $u_i = u_i^0 + \tilde{u}_i$ for $i = 1, 2$

$$\begin{aligned} \nu \nabla \times \nabla \times u_i^0 &= f \quad \text{in } \Omega_i \\ \nabla \cdot u_i^0 &= 0 \quad \text{in } \Omega_i \\ n \times u_i^0 &= 0 \quad \text{on } \partial\Omega_i \\ n \times u_i^0 &= 0 \quad \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} \nu \nabla \times \nabla \times \tilde{u}_i &= 0 \quad \text{in } \Omega_i \\ \nabla \cdot \tilde{u}_i &= 0 \quad \text{in } \Omega_i \\ n \times \tilde{u}_i &= 0 \quad \text{on } \partial\Omega_i \\ n \times \tilde{u}_i &= \varphi \quad \text{on } \Gamma \end{aligned}$$

The first transmission condition is ensured, we must then ensure the second one:

$$\underbrace{\nu_1(\nabla\times\tilde{u}_i\times n)-\nu_2(\nabla\times\tilde{u}_2\times n)}_{S\varphi}=\underbrace{-\nu_1(\nabla\times u_i^0\times n)+\nu_2(\nabla\times u_2^0\times n)}_{g\ (\text{Known})}$$

where S is the Steklov-Poincaré operator

We must find $\varphi\in H^{1/2}(\partial\Omega)$ such that the transmission condition is ensured.

c.) Discrete version of the previous scheme.

- Ω_1 (Neumann problem)

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix}\begin{bmatrix} u_1^k \\ u_\Gamma^k \end{bmatrix}=\begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}u_2^{k-1}-A_{\Gamma\Gamma}^{(2)}u_\Gamma^{k-1}\end{bmatrix}$$

- Ω_2 (Dirichlet problem)

$$A_{22}u_2^k=F_2-A_{2\Gamma}u_\Gamma^l \qquad l=\begin{cases} k-1 & (\text{Jacobi}) \\ k & (\text{Gauss Seidel})\end{cases}$$

3.) Laplace eq.

$$u:\Omega\longrightarrow\mathbb{R} \qquad -k\Delta u=f \text{ in } \Omega$$
$$u=0 \text{ on } \partial\Omega$$

a.) Iteration by subdomain based on Dirichlet-Robin coupling.

$$\begin{aligned} -k_1\Delta u_1^k &= f && \text{in } \Omega_1 \\ u_1^k &= 0 && \text{on } \partial\Omega_1 \\ k_1\frac{\partial u_1^k}{\partial n}+\gamma_1 u_1^k &= k_2 u_2^{k-1}+\gamma_2 u_2^{k-1} && \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} -k_2\Delta u_2^k &= f && \text{in } \Omega_2 \\ u_2^k &= 0 && \text{on } \partial\Omega_2 \\ u_2^k &= u_1^l && \text{on } \Gamma \end{aligned}$$

b.) Matrix version of discretized version of previous scheme

Let us first obtain the weak form of the problem

$$-\int_\Omega \delta u(k\Delta u)=\int_\Omega \delta u\, f$$

$$-\int_{\partial\Omega} \delta u \left(K \frac{\partial u}{\partial n}\right) + \int_{\Omega} \nabla \delta u \,(K \nabla u) = \int_{\Omega} \delta u\, f$$

$$\int_{\Omega} \nabla \delta u\,(K \nabla u) = \int_{\Omega} \delta u f + \int_{\partial\Omega} \delta u \left(K \frac{\partial u}{\partial n}\right)$$

- Ω_1 (Robin problem)

  BCs are imposed weakly

$$\int_{\Omega_1} \nabla \delta u (K \nabla u) + \int_{\Gamma} \delta u (\gamma_1 u_1) = \int_{\Omega_1} \delta u f + \int_{\Gamma} \delta u \left(K_2 \frac{\partial u_2}{\partial n}\right) + \int_{\Gamma} \delta u (\gamma_2 u_2)$$

- Ω_2 (Dirichlet Problem)

  BCs are imposed strongly

$$\int_{\Omega_2} \nabla \delta u (K \nabla u) = \int_{\Omega_2} \delta u f$$

After applying FE discretization:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + M_{\Gamma\Gamma}^{(1)} \gamma_1 \end{bmatrix} \begin{bmatrix} u_1^k \\ u_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - (A_{\Gamma 2} - M_{\Gamma 2} \gamma_2) u_2^{k-1} - (A_{\Gamma\Gamma}^{(2)} - \gamma_2 M_{\Gamma\Gamma}^{(2)}) u_\Gamma^{k-1} \end{bmatrix}$$

$$A_{22} u_2^k = F_2 - A_{2\Gamma} u_\Gamma^{k}$$

c.) Schur Complement as discrete version of Steklov - Poincaré operator.

$$-K \Delta u_i^0 = f \quad \text{in } \Omega_i \qquad\qquad -K \Delta \tilde{u}_i = 0 \quad \text{in } \Omega_i$$

$$u_i^0 = 0 \quad \text{on } \partial\Omega_i \qquad\qquad \tilde{u}_i = 0 \quad \text{on } \partial\Omega_i$$

$$u_i^0 = 0 \quad \text{on } \Gamma \qquad\qquad \tilde{u}_i = \varphi \quad \text{on } \Gamma$$

$$K \frac{\partial u_1}{\partial n} = K_2 \frac{\partial u_2}{\partial n} \longrightarrow K_1 \frac{\partial \tilde{u}_1}{\partial n} - K_2 \frac{\partial \tilde{u}_2}{\partial n} = \underbrace{-K_1 \frac{\partial u_1^0}{\partial n} + K_2 \frac{\partial u_2^0}{\partial n}}_{G}$$

$$u_1^0 = A_{11}^{-1} F_1 \qquad\qquad \tilde{u}_1 = -A_{11}^{-1} A_{1\Gamma} u_\Gamma$$

$$u_2^0 = A_{22}^{-1} F_2 \qquad\qquad \tilde{u}_2 = -A_{22}^{-1} A_{2\Gamma} u_\Gamma$$

$$G = F_\Gamma - A_{\Gamma 1} U_1^0 - A_{\Gamma 2} U_2^0 = \underbrace{A_{\Gamma\Gamma} U_\Gamma + A_{\Gamma 1} \tilde{U}_1 + A_{\Gamma 2} \tilde{U}_2}_{S U_\Gamma}$$

$$\underbrace{\left(A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right)}_{S} U_\Gamma = F_\Gamma - A_{\Gamma 1} U_1^0 - A_{\Gamma 2} U_2^0$$

d.) Preconditioner for Schur complement equation arising from the Iterative scheme of section (a)

$$S = S_1 + S_2 \qquad (S_1 + S_2) U_\Gamma = G$$

$$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} \qquad S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

Considering a Gauss-Seidel-type iteration by subdomain:

$$U_1^k = A_{11}^{-1} (F_1 - A_{1\Gamma} U_\Gamma^k) \qquad U_2^k = A_{22}^{-1} (F_2 - A_{2\Gamma} U_\Gamma^k)$$

$$A_{\Gamma 1} U_1^k + \left(A_{\Gamma\Gamma}^{(1)} + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^k = F_\Gamma - (A_{\Gamma 2} - \gamma_2 M_{\Gamma 2}) U_2^{k-1} - \left(A_{\Gamma\Gamma}^{(2)} - \gamma_2 M_{\Gamma\Gamma}^{(2)}\right) U_\Gamma^{k-1}$$

$$A_{\Gamma 1} A_{11}^{-1} (F_1 - A_{1\Gamma} U_\Gamma^k) + \left(A_{\Gamma\Gamma}^{(1)} + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^k = F_\Gamma - (A_{\Gamma 2} - \gamma_2 M_{\Gamma 2}) A_{22}^{-1} (F_2 - A_{2\Gamma} U_\Gamma^{k-1}) - \left(A_{\Gamma\Gamma}^{(2)} - \gamma_2 M_{\Gamma\Gamma}^{(2)}\right) U_\Gamma^{k-1}$$

$$\left(S_1 + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^k + A_{\Gamma 1} A_{11}^{-1} F_1 = F_\Gamma - \left(S_2 + \gamma_2 M_{\Gamma\Gamma}^{(2)}\right) U_\Gamma^{k-1} - (A_{\Gamma 2} - \gamma_2 M_{\Gamma 2}) A_{22}^{-1} F_2 - \gamma_2 M_{\Gamma 2} A_{22}^{-1} A_{2\Gamma} U_\Gamma^{k-1}$$

$$\left(S_1 + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^k + A_{\Gamma 1} \underbrace{A_{11}^{-1} F_1}_{U_1^0} = F_\Gamma - \left(S_2 + \gamma_2 M_{\Gamma\Gamma}^{(2)} + \gamma_2 M_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right) U_\Gamma^{k-1} - (A_{\Gamma 2} - \gamma_2 M_{\Gamma 2}) \underbrace{A_{22}^{-1} F_2}_{U_2^0}$$

$$\left(S_1 + \gamma M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^k = G - (S + \gamma M_{\Gamma\Gamma}) U_\Gamma^{k-1} + \left(S_1 + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right) U_\Gamma^{k-1}$$

$$\boxed{U_\Gamma^k = U_\Gamma^{k-1} + \left(S_1 + \gamma_1 M_{\Gamma\Gamma}^{(1)}\right)^{-1} \left(G - (S + \gamma M_{\Gamma\Gamma}) U_\Gamma^{k-1}\right)}$$

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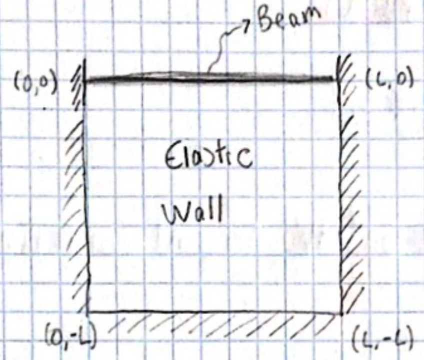
HW 3

Coupling of Heterogeneous Problems

1.) Euler - Bernoulli Beam



- wall displacements in x an y are u and v
- young's modulus E and poisson coef. ν
- Only loads on the wall are those coming from the beam

a.) Write down the equations in the wall assuming plane stress behavior

The problem of the wall is governed by the Cauchy momentum equation:

$$\nabla \cdot \sigma = \underset{\text{no body forces}}{0} \quad \text{in } \Omega_{wall}$$

$$\begin{cases} \frac{C}{2} \nabla \cdot (\nabla \underline{u} + (\nabla \underline{u})^T) = 0 & \text{in } \Omega_{wall} \\ \underline{u} = 0 & \text{on } \Gamma_D \end{cases}$$

$$\sigma = C\varepsilon = C(\nabla^s u) = C \cdot \frac{1}{2}(\nabla u + (\nabla u)^T)$$

For plane stress: (voigt notation)

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$\Gamma_D \rightarrow$ clamped walls

→ Transmission conditions must be enforced at the upper boundary where the wall is in contact with the beam.

b.) Write down the modified equations for the beam

Euler - Bernoulli beam theory

$$\begin{cases} EI\frac{d^4 v}{dx^4} = f(x) - R_y(x) & \text{in } \Omega_{beam} \\ v(0) = v(L) = 0 \\ \frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0 \end{cases}$$

$R_y(x) \rightarrow$ wall reaction

c.) Adequate transmission conditions for v and the normal component of the traction on the wall at $y=0$

We must have continuity of displacements at the contact curve Γ_{12}

$$[\![v]\!]_{\Gamma_{12}} = 0 \longrightarrow \text{strongly enforced}$$

The normal stress at Γ_{12} must be continuous

$$[\![n\cdot(n\cdot\sigma)]\!]_{\Gamma_{12}} = 0 \longrightarrow \text{weakly enforced}$$

d.) Suggest transmission conditions for u and the tangent component of the traction on the wall at $y=0$

The differential equation derived from the Euler-Bernoulli theory describes the vertical displacement of a beam's neutral axis. It does not consider axial stress in the beam and therefore, it does not consider axial (horizontal) displacement.

Therefore, for the Euler-Bernoulli equation to be satisfied, the horizontal displacements and tangential stress of the wall at Γ would have to be zero.

$$u(x,0) = 0$$

$$\tau_{xy}(x,0) = 0$$

2.) Let S_D and S_S be the Dirichlet-to-Neumann operators for Darcy and Stokes Problems. Steklov Poincaré eq:

$$S_S(\lambda) = S_D(\lambda)$$

$\lambda \rightarrow$ normal velocity on Γ

$\Gamma \rightarrow$ interface between Darcy and Stokes Regions

a.) Discrete version of the eq. after FE spatial discretization

Stokes problem:

$$-\nu\Delta u_S + \nabla p_S = f \quad \text{in } \Omega_S$$

$$\nabla\cdot u_S = 0 \quad \text{in } \Omega_S$$

$$u_S = \bar{u}_S \quad \text{on } \Gamma_S$$

Darcy problem:

$$u_D + K\nabla\varphi = 0 \quad \text{in } \Omega_D$$

$$\nabla\cdot u_D = 0 \quad \text{in } \Omega_D$$

$$n\cdot u_S = \bar{u}_{n,D} \quad \text{on } \Gamma_D$$

Transmission Conditions:

$$n \cdot u_s = n \cdot u_D \longrightarrow \text{Applied strongly}$$

$$\left.\begin{aligned} & p_s - \nu\, n \cdot \nabla u_s \cdot n = \varphi \\ & u_s \cdot t + \frac{\sqrt{K}}{\alpha_{BJ}} \nu\, t \cdot (n \cdot \nabla u_s) = 0 \end{aligned}\right\} \longrightarrow \text{Applied weakly}$$

Let us recall the weak form of the problems:

- Stokes:

$$\int_{\Omega_s} \nu \nabla \delta u_s : \nabla u_s - \int_{\Omega_s} p_s \nabla \cdot \delta u_s = \int_{\Omega_s} \delta u f + \int_{\partial\Omega_s} \delta u_s \cdot [n \cdot (-p_s I + \nu \nabla u_s)]$$

Replacing $\delta u = N$ and $u = \sum N_i u_i$

and defining:

$$A = \nu \int_{\Omega_s} \nabla N_i \nabla N_j \qquad G = -\int N_i \nabla \cdot N_j \longrightarrow \begin{bmatrix} A & G \\ G & 0 \end{bmatrix} \begin{bmatrix} u_s \\ p_s \end{bmatrix} = \begin{bmatrix} F_s \\ 0 \end{bmatrix}$$

- Darcy:

$$\int_{\Omega_D} K^{-1} \delta u_D \cdot u_D - \int_{\Omega_D} \varphi \nabla \cdot \delta u_D = \int_{\partial\Omega_D} \varphi\, n \cdot \delta u_D$$

$$M = K^{-1} \int N_i N_j \qquad \begin{bmatrix} M & G \\ G & 0 \end{bmatrix} \begin{bmatrix} u_D \\ \varphi \end{bmatrix} = \begin{bmatrix} F_D \\ 0 \end{bmatrix}$$

Rewritting both systems with $u = [u_I, \lambda]^T$:

$$\begin{bmatrix} A^s_{II} & A^s_{I\Gamma} & B^s_{II} \\ A^s_{\Gamma I} & A^s_{\Gamma\Gamma} & B^s_{\Gamma I} \\ B^s_{II} & B^s_{I\Gamma} & 0 \end{bmatrix} \begin{bmatrix} u^s_I \\ \lambda \\ p^s \end{bmatrix} = \begin{bmatrix} F^s_I \\ F^s_\Gamma \\ 0 \end{bmatrix} \qquad \begin{bmatrix} A^D_{II} & A^D_{I\Gamma} & B^D_{II} \\ A^D_{\Gamma I} & A^D_{\Gamma\Gamma} & B^D_{\Gamma I} \\ B^D_{II} & B^D_{I\Gamma} & 0 \end{bmatrix} \begin{bmatrix} u^D_I \\ u^D_\Gamma \\ \varphi \end{bmatrix} = \begin{bmatrix} F^D_I \\ F^D_\Gamma \\ 0 \end{bmatrix}$$

(Stokes) (Darcy)

Joining both systems of equations:

$$\begin{bmatrix} A_{II}^S & B_{II}^S & A_{I\Gamma}^S & 0 & 0 \\ B_{II}^S & 0 & B_{I\Gamma}^S & 0 & 0 \\ A_{\Gamma I}^S & B_{\Gamma I}^S & A_{\Gamma\Gamma} & A_{\Gamma I}^D & B_{\Gamma I}^D \\ 0 & 0 & B_{I\Gamma}^D & 0 & B_{II}^D \\ 0 & 0 & A_{I\Gamma}^D & B_{II}^D & A_{II}^D \end{bmatrix} \begin{bmatrix} u_I^S \\ p^S \\ \lambda \\ P_D \\ u_I^D \end{bmatrix} = \begin{bmatrix} F_I^S \\ 0 \\ F_\Gamma \\ 0 \\ F_I^D \end{bmatrix}$$

$$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^S + A_{\Gamma\Gamma}^D$$

$$F_\Gamma = F_\Gamma^S + F_\Gamma^D$$

Let us condense the matrices in the following way

$$A_{SS} = \begin{bmatrix} A_{II}^S & B_{II}^S \\ B_{II}^S & 0 \end{bmatrix} \qquad A_{S\Gamma} = \begin{bmatrix} A_{I\Gamma}^S \\ B_{I\Gamma}^S \end{bmatrix} \qquad A_{\Gamma\Gamma} = A_{\Gamma\Gamma}$$

The system of equations takes the form:

$$\begin{bmatrix} A_{SS} & A_{S\Gamma} & 0 \\ A_{\Gamma S} & A_{\Gamma\Gamma} & A_{\Gamma D} \\ 0 & A_{D\Gamma} & A_{DD} \end{bmatrix} = \begin{bmatrix} u_S \\ \lambda \\ u_D \end{bmatrix} = \begin{bmatrix} F_S \\ F_\Gamma \\ F_D \end{bmatrix} \longrightarrow \begin{array}{l} u_S = A_{SS}^{-1}(F_S - A_{S\Gamma}\lambda) \\ u_D = A_{DD}^{-1}(F_D - A_{D\Gamma}\lambda) \end{array}$$

Replacing u_S and u_D in the second equation:

$$A_{\Gamma S} A_{SS}^{-1}(F_S - A_{S\Gamma}\lambda) + A_{\Gamma\Gamma}\lambda + A_{\Gamma D} A_{DD}^{-1}(F_D - A_{D\Gamma}\lambda) = F_\Gamma$$

$$\underbrace{(A_{\Gamma\Gamma} - A_{\Gamma S} A_{SS}^{-1} A_{S\Gamma} - A_{\Gamma D} A_{DD}^{-1} A_{D\Gamma})}_{S = S_S + S_D}\lambda = \underbrace{F_\Gamma - A_{\Gamma S} A_{SS}^{-1} F_S - A_{\Gamma D} A_{DD}^{-1} F_D}_{G}$$

b.) Matrix form of DN iteration-by-subdomain scheme

$$\begin{bmatrix} A_{SS} & A_{S\Gamma} \\ A_{\Gamma S} & A_{\Gamma\Gamma}^{(S)} \end{bmatrix} \begin{bmatrix} u_S^k \\ \lambda^k \end{bmatrix} = \begin{bmatrix} F_S \\ F_\Gamma - A_{\Gamma\Gamma}^{(D)}\lambda^{k-1} - A_{\Gamma D} u_D^{k-1} \end{bmatrix}$$

$$A_{DD} u_D^k = F_D - A_{D\Gamma}\lambda^k$$

c.) Richardson iteration for the algebraic problem in (a) resulting from (b)

From the previous DN scheme we know:

$$A_{\Gamma S} U_S^k + A_{\Gamma\Gamma}^{(S)} \lambda^k = F_\Gamma - A_{\Gamma\Gamma}^{(D)} \lambda^{k-1} - A_{\Gamma D} U_D^{k-1} \quad ①$$

$$U_S^k = A_{SS}^{-1}(F_S - A_{S\Gamma}\lambda^k) \quad ②$$

$$U_D^{k-1} = A_{DD}^{-1}(F_D - A_{D\Gamma}\lambda^{k-1}) \quad ③$$

Replacing ② and ③ into ① yields:

$$A_{\Gamma S} A_{SS}^{-1}(F_S - A_{S\Gamma}\lambda^k) + A_{\Gamma\Gamma}^{(S)}\lambda^k = F_\Gamma - A_{\Gamma\Gamma}^{(D)}\lambda^{k-1} - A_{\Gamma D}A_{DD}^{-1}(F_D - A_{D\Gamma}\lambda^{k-1})$$

$$\underbrace{(A_{\Gamma\Gamma}^{(S)} - A_{\Gamma S}A_{SS}^{-1}A_{S\Gamma})}_{S_S}\lambda^k + A_{\Gamma S}A_{SS}^{-1}F_S = F_\Gamma - \underbrace{(A_{\Gamma\Gamma}^{(D)} - A_{\Gamma D}A_{DD}^{-1}A_{D\Gamma})}_{S_D}\lambda^{k-1} - A_{\Gamma D}A_{DD}^{-1}F_D$$

$$S_S\lambda^k = F_\Gamma - S\lambda^{k-1} + S_S\lambda^{k-1} - A_{\Gamma D}A_{DD}^{-1}F_D - A_{\Gamma S}A_{SS}^{-1}F_S$$

$$\boxed{\lambda^k = \lambda^{k-1} + S_S^{-1}(G - S\lambda^{k-1})}$$

Richardson precond. $P = S_S$

HW 4

Monolithic and Partitioned schemes in time

One dimensional, transient, heat transfer equation

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1]$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = 0$$

1.) Discretize using the finite element method (linear elements of size h) for the discretization in space, and a BDF1 scheme for time, weak form of the problem and resulting matrix form, including boundary integrals.

$$K=1 \quad f=1 \quad \delta t = 1$$

- Weak form:

$$\int_\Omega w \cdot u_t - \int_\Omega w \cdot K u_{xx} = \int_\Omega w \cdot f$$

Integrating by parts:

$$\int_\Omega w \cdot u_t + \int_\Omega \nabla w \cdot K \nabla u - \cancel{\int_{\partial\Omega} w \cdot K \nabla u}^{\,w=0 \text{ at } \partial\Omega} = \int_\Omega w f \qquad ①$$

Replacing:

$$w = N \qquad u \approx u^h = \sum N_i(x)\, u_i(t) \qquad u_t \approx \sum N_i(x) \frac{du_i(t)}{dt}$$

① becomes

$$\underbrace{\left(\int_\Omega N_i N_j\right)}_{\substack{M \\ \text{mass matrix}}} \frac{du_j}{dt} + \underbrace{\left(\int_\Omega \nabla N_i \cdot \overset{K=1}{K} \nabla N_j\right)}_{\substack{K \\ \text{stiffness matrix}}} u_j = \underbrace{\int_\Omega N_i f}_{\substack{F \\ \text{forces vector}}}$$

Rewritting the equation replacing the matrices yields:

$$\boxed{M\frac{du}{dt} + Ku = F} \quad (2)$$

Using a BDF1 scheme to discretize in time:

$$\frac{\partial u^{n+1}}{\partial t} = \frac{(u^{n+1} - u^n)}{\Delta t}$$

Replacing in (2):

$$M\frac{(u^{n+1} - u^n)}{\Delta t} + Ku^{n+1} = F^{n+1}$$

Given that $f=1$ and $\Delta t = 1$:

$$(M+K)u^{n+1} = F + Mu^n$$

$$\boxed{u^{n+1} = (M+K)^{-1}(F + Mu^n)}$$

2.) Domain decomposition approach of previous problem. Left subdomain of two elements ($h=0,2$). Right subdomain of three elements ($h=0,2$).

Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface

$$u = [u_0 \ u_1 \ u_2 \ u_3 \ u_4 \ u_5]^T \qquad u_\Gamma = u_2$$

The weak form at each subdomain takes the form:

$$\int_{\Omega_1} N_i N_j \frac{du}{dt} + \int_{\Omega_1} \nabla N_i \cdot \nabla N_j \, u - \int_{\Gamma} N_i \nabla N_j \, u_1 = \int_{\Omega_1} N_i f \qquad \Omega_1 = [0, 0.4]$$

$$\int_{\Omega_2} N_i N_j \frac{du}{dt} + \int_{\Omega_2} \nabla N_i \cdot \nabla N_j \, u - \int_{\Gamma} N_i \nabla N_j \, u_2 = \int_{\Omega_2} N_i f \qquad \Omega_2 = (0.4, 1]$$

Considering the transmission conditions:

$$[\![u]\!] = 0 \quad \text{temperature continuity}$$

$$[\![\nabla u]\!] = 0 \quad \text{fluxes continuity}$$

Adding both subdomains

$$\int_0^{0,4} N_i N_j \, dx \frac{du}{dt} + \int_{0,4}^{1} N_i N_j \, dx \frac{du}{dt} + \int_0^{0,4} \nabla N_i \nabla N_j \, dx \, u + \int_{0,4}^{1} \nabla N_i \nabla N_j \, dx \, u - \cancel{[N_i \nabla N_j]}_{x=0,4^+} u_1$$

$$- \cancel{[N_i \nabla N_j]}_{x=0,4^-} u_2 = \int_0^{0,4} N_i f \, dx + \int_{0,4}^{1} N_i f \, dx$$

$$\boxed{\int_0^1 N_i N_j \, dx \frac{du}{dt} + \int_0^1 \nabla N_i \nabla N_j \, dx \, u = \int_0^1 N_i f \, dx} \longrightarrow \text{original problem}$$

No integrals are required at the interface

3) Algebraic form of Dirichlet-to-Neumann operator (Steklov-Poincaré operator) for the left subdomain, departing from given values of u_i^n and an interface value u_2^{n+1}.

Defining $\underline{A = M + K}$

$$u^{n+1} = A^{-1}(F + M u^n)$$

Left subdomain $\rightarrow$ $u^{(1)} = [\underbrace{u_0 \;\; u_1}_{u_i^{(1)}} \;\; \underbrace{u_2}_{u_\Gamma}]^T$

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} M_{00} & M_{01} & M_{02} \\ M_{10} & M_{11} & M_{12} \\ M_{20} & M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_0^n \\ u_1^n \\ u_2^n \end{bmatrix}$$

Applying Dirichlet BC $\rightarrow u_0(t) = 0$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 + M_{11} u_1^n + M_{12} u_2^n \\ F_2 + M_{21} u_1^n + M_{22}^{(1)} u_2^n + \phi \end{bmatrix}$$

$\hookrightarrow$ fluxes from subdomain 2

u_2^{n+1} is also applied as a Dirichlet BC on subdomain 1.

Therefore, from equation 1:

$$\boxed{A_{11} u_1^{n+1} = \left(F_1 + M_{11} u_1^n + M_{12} u_2^n - A_{12} u_2^{n+1}\right)}$$

4.) Algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = K \frac{\partial u^{n+1}}{\partial x}$ at node 2.

Right subdomain $\longrightarrow u^{(2)} = [u_2 \; u_3 \; u_4 \; u_5]$

Applying Dirichlet BC $u_5(t) = 0$:

$$\begin{bmatrix} A_{22}^{(2)} & A_{23} & \cancel{A_{24}}^{\,0} \\ A_{32} & A_{33} & A_{34} \\ \cancel{A_{42}}_{\,0} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \end{bmatrix} = \begin{bmatrix} F_2 + \overset{\text{Neumann BC}}{\phi^{n+1}} + M_{22} u_2^n + M_{23} u_3^n + \cancel{M_{24} u_4^n}^{\,0} \\ F_3 + M_{32} u_2^n + M_{33} u_3^n + M_{34} u_4^n \\ F_4 + \cancel{M_{42} u_2^n}^{\,0} + M_{43} u_3^n + M_{44} u_4^n \end{bmatrix}$$

5.) Iterative algorithm for staggered approach applying Dirichlet BCs at the interface to the left subdomain and Neumann BCs at the interface for the right subdomain

First order approximation $\longrightarrow \underline{\underline{\tilde{u}^{n+1} = u^n}}$

- Subdomain 1: (Dirichlet)

$$A_{11} u^{n+1,i} = F_1 + M_{11} u_1^n + M_{12} u_2^n - A_{12} u_2^{n+1,i-1}$$

- Subdomain 2 (Neumann)

$$\begin{bmatrix} A_{22}^{(2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} u_2^{n+1,i} \\ u_3^{n+1,i} \\ u_4^{n+1,i} \end{bmatrix} = \begin{bmatrix} F_2 + M_{22} u_2^n + M_{23} u_3^n - \underline{\underline{A_{21} u_1^{n+1,i-1}}} - \underline{\underline{A_{22}^{(1)} u_2^{n+1,i-1}}} \\ F_3 + M_{32} u_2^n + M_{33} u_3^n + M_{34} u_4^n \\ F_4 + M_{43} u_3^n + M_{44} u_4^n \end{bmatrix}$$

6.) Same for a substitution and iteration-by-subdomain scheme.

Initial guess at subdomain 2 $\longrightarrow \tilde{u}^{n+1} = u^n$

Solve subdomain 1 using the results obtained at subdomain 2

- Subdomain 2:

$$\begin{bmatrix} A_{22}^{(2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} u_2^{n+1,i} \\ u_3^{n+1,i} \\ u_4^{n+1,i} \end{bmatrix} = \begin{bmatrix} F_2 + M_{22}u_2^n + M_{23}u_3^n - \underline{\underline{A_{21}u_1^{n+1,i-1}}} - \underline{\underline{A_{22}^{(1)}u_2^{n+1,i-1}}} \\ F_3 + M_{32}u_2^n + M_{33}u_3^n + M_{34}u_4^n \\ F_4 + M_{43}u_3^n + M_{44}u_4^n \end{bmatrix}$$

Solve for $u_2^{n+1,i}$ and apply to subdomain 1

- Subdomain 1

$$A_{11}u_1^{n+1,i} = F_1 + M_{11}u_1^n + M_{12}u_2^n - A_{12}u_2^{n+1,i}$$

7.) Rewrite the algebraic system associated to the left subdomain, using Nitsche's method for applying BC's. How does the condition number of the resulting system of equations vary with the penalty parameter α?

Let us recall the weak form of the problem:

$$\left(w, \frac{\partial u}{\partial t}\right) + (\nabla w, \nabla u) - \langle w, n\cdot\nabla u\rangle_\Gamma = \langle w, f\rangle$$

Adding the terms corresponding to the Nitsche method:

$$\left(w, \frac{\partial u}{\partial t}\right) + (\nabla w, \nabla u) - \langle w, n\cdot\nabla u\rangle_\Gamma + \underbrace{\frac{\alpha}{h}(w,u)_\Gamma - \langle n\cdot\nabla w, u\rangle_\Gamma}_{N} = \langle w, f\rangle + \frac{\alpha}{h}(w,\bar{u})_\Gamma - \langle n\cdot\nabla w, \bar{u}\rangle_\Gamma$$

using FE discretization

and defining $N = \frac{\alpha}{h}\int_\Gamma N_i N_j \, d\Gamma - \int_\Gamma \nabla N_i \cdot N_j \, d\Gamma$ $\qquad F_N = \frac{\alpha}{h}\int_\Gamma N_i \bar{u} - [\nabla N_i \bar{u}]_\Gamma$

$$M\frac{du}{dt} + Ku + Nu = F + F_N$$

Applying BDF1 scheme for time discretization and considering $\Delta t = 1$

$$\boxed{(M + U + N)\,u^{n+1} = F + F_N + M u^n}$$

The condition number of the system grows as α grows. For a large value of α, the system might become ill conditioned. However, using Nitsche's method guarantees stability for $\alpha > 2C_i$, therefore allowing us to avoid large values of α which would cause problems.

HW 5

Operator Splitting Techniques

One-dimensional, transient, convection-diffusion eq.

$$\begin{cases} \dfrac{\partial u}{\partial t} - K\dfrac{\partial^2 u}{\partial x^2} + a_x \dfrac{\partial u}{\partial x} = f \quad \text{in } [0,1] \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 0 \end{cases} \qquad \begin{cases} K = 1 \\ a_x = 1 \\ f = 1 \end{cases}$$

1.) Space discretization with 3 FE. Time discretization with BDF1
Solve the first step.

$$\int_\Omega w\, u_t - \int_\Omega w \nabla^2 u + \int_\Omega w \nabla u = \int_\Omega w f$$

Integrating by parts:

$$\int_\Omega w\, u_t - \cancel{\int_{\partial\Omega} w \nabla u}_{w=0} + \int_\Omega \nabla w \nabla u + \int_\Omega w \nabla u = \int_\Omega w f$$

- FE discretization:

$$w^h = N(x) \qquad u^h = \sum N_i(x)\, u_i(t) \qquad \frac{\partial u^h}{\partial t} = \sum N_i(x) \frac{du_i}{dt}$$

$$M = \int_\Omega N_i N_j \qquad K = \int_\Omega \nabla N_i \nabla N_j \qquad C = \int_\Omega N_i \nabla N_j \qquad F = \int_\Omega N_i f$$

$$M \frac{du}{dt} + K u + C u = F$$

- Time discretization (BDF 1)

$$\frac{1}{\Delta t} M u^{n+1} + K u^{n+1} + C u^{n+1} = F^{n+1} + \frac{1}{\Delta t} M u^n$$

Interpolating functions $\longrightarrow N_1 = \dfrac{x_2 - x}{h} \qquad N_2 = \dfrac{x - x_1}{h}$

$$\frac{dN_1}{dx} = -\frac{1}{h} \qquad \frac{dN_2}{dx} = \frac{1}{h} \qquad h: \text{element length}$$

$$h = \frac{1}{3}$$

$$M^{(e)} = \int_{\Omega^e} N_i N_j = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix}$$

$$\int_{x_1}^{x_2} N_1^2 = \frac{1}{h^2} \int_{x_1}^{x_2} (x_2 - x)^2 = \frac{1}{h^2}\left[x_2^2 x - x_2 x^2 + \frac{x^3}{3} \right]_{x_1}^{x_2}$$

$$= \frac{1}{h^2}\left(\cancel{x_2^3} - \cancel{x_2^3} + \frac{x_2^3}{3} - x_2^2 x_1 + x_2 x_1^2 - \frac{x_1^3}{3} \right) = \frac{1}{h^2}\left(\frac{x_2^3 - x_1^3}{3} + (x_2 x_1^2 - x_2^2 x_1) \right)$$

$$\int_{x_1}^{x_2} N_2^2 = \frac{1}{h^2} \int_{x_1}^{x_2} (x - x_1)^2 = \frac{1}{h^2}\left[x_1^2 x - x_1 x^2 + \frac{x^3}{3} \right]_{x_1}^{x_2}$$

$$= \frac{1}{h^2}\left[x_1^2 x_2 - x_1 x_2^2 + \frac{x_2^3}{3} - \cancel{x_1^3} + \cancel{x_1^3} - \frac{x_1^3}{3} \right] = \frac{1}{h^2}\left[\frac{x_2^3 - x_1^3}{3} + x_2 x_1^2 - x_2^2 x_1 \right]$$

$$\int_{x_1}^{x_2} N_1 N_2 = \frac{1}{h^2} \int_{x_1}^{x_2} (x_2 - x)(x - x_1) = \frac{1}{h^2}\left[\frac{x^2}{2}(x_1 + x_2) - x_1 x_2 x - \frac{x^3}{3} \right]_{x_1}^{x_2}$$

$$= \frac{1}{h^2}\left[\frac{x_2^2}{2}(x_1 + x_2) - x_1 x_2^2 - \frac{x_2^3}{3} - \frac{x_1^2}{2}(x_1 + x_2) + x_1^2 x_2 + \frac{x_1^3}{3} \right]$$

$$= \frac{1}{h^2}\left[\frac{x_1 x_2^2 + x_2^3 - x_1^3 - x_1^2 x_2}{2} + \frac{x_1^3 - x_2^3}{3} + x_1^2 x_2 - x_1 x_2^2 \right]$$

$$= \frac{1}{h^2}\left[\frac{x_1^2 x_2 - x_1 x_2^2 + x_2^3 - x_1^3}{2} + \frac{x_1^3 - x_2^3}{3} \right]$$

$$\text{Local} \longrightarrow M^{(e)} = \frac{1}{9}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$\text{Global} \longrightarrow M = \frac{1}{9}\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 2 & 1/2 & 0 \\ 0 & 1/2 & 2 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$K^{(e)} = \int_{\Omega_e} \nabla N_i \nabla N_j = \frac{1}{h^2}\int_{x_1}^{x_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\text{Global} \longrightarrow K = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$C^{(e)} = \int_{\Omega_e} N_i \nabla N_j = \frac{1}{h^2}\int_{x_1}^{x_2} \begin{bmatrix} (x-x_2) & (x_2-x) \\ (x_1-x) & (x-x_1) \end{bmatrix} = \frac{1}{h^2}\begin{bmatrix} (\frac{x^2}{2} - x_2 x) & (x_2 x - \frac{x^2}{2}) \\ (x_1 x - \frac{x^2}{2}) & (\frac{x^2}{2} - x_1 x) \end{bmatrix}_{x_1}^{x_2}$$

$$C^{(e)} = 9\begin{bmatrix} -\frac{1}{18} & \frac{1}{18} \\ -\frac{1}{18} & \frac{1}{18} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\text{Global} \longrightarrow C = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\left(\underbrace{\frac{1}{9\Delta t}\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ & 2 & \frac{1}{2} & 0 \\ & & 2 & \frac{1}{2} \\ \text{sym} & & & 1 \end{bmatrix}}_{\frac{1}{\Delta t}M} + \underbrace{\begin{bmatrix} 3 & -3 & 0 & 0 \\ & 6 & -3 & 0 \\ & & 6 & -3 \\ \text{sym} & & & 3 \end{bmatrix}}_{K} + \frac{1}{2}\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{C}\right)\begin{bmatrix} 0 \\ u_2^{n+1} \\ u_3^{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} + \frac{MU^n}{\Delta t}$$

Imposing Dirichlet BCs and initial conditions:

$$\underbrace{\left(\frac{1}{9\Delta t}\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)}_{A}\begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + \cancel{\frac{1}{9\Delta t}\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} u_2^1 \\ u_3^1 \end{bmatrix} = A^{-1} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

Computing the solution directly with the aid of computational tools:

$$\boxed{\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{6\Delta t}{2943\Delta t^2 + 324\Delta t + 5} \begin{bmatrix} 51\Delta t + 1 \\ 57\Delta t + 1 \end{bmatrix}}$$

2.) Solve some time step using 1st order operator splitting technique.

$$\frac{\partial u}{\partial t} + \mathcal{L}_a u + \mathcal{L}_v u = f \qquad \mathcal{L}_a u = \underset{1}{a} \cdot \nabla u \qquad \mathcal{L}_v u = -\underset{1}{K} \nabla^2 u$$

- 1st step:

$$u_a^n = u^n$$

$$\frac{\partial u_a}{\partial t} + \mathcal{L}_a u_a = 0 \longrightarrow \frac{u_a^{n+1} - u^n}{\Delta t} + \mathcal{L}_a u_a^{n+1} = 0$$

$$\left(\frac{1}{\Delta t} M + C\right) u_a^{n+1} = \frac{1}{\Delta t} M u_a^n = \frac{1}{\Delta t} M u^n$$

initial conditions

$$\left(\frac{1}{9\Delta t}\begin{bmatrix} 2 & 1/2 \\ 1/2 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\begin{bmatrix} u_{a_2}^{n+1} \\ u_{a_3}^{n+1} \end{bmatrix} = \frac{1}{\Delta t} M \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\underline{u_a^{n+1}} = \underline{0}}$$

- 2nd step

$$u_v^n = u_a^{n+1}$$

$$\frac{\partial u_v}{\partial t} + \mathcal{L}_v u_v = f \longrightarrow \frac{u_v^{n+1} - u_v^n}{\Delta t} + \mathcal{L}_v u_v^{n+1} = f$$

$$u^{n+1} = u_v^{n+1}$$

$$\frac{u^{n+1} - u_a^{n+1}}{\Delta t} + \mathcal{L}_v u^{n+1} = f$$

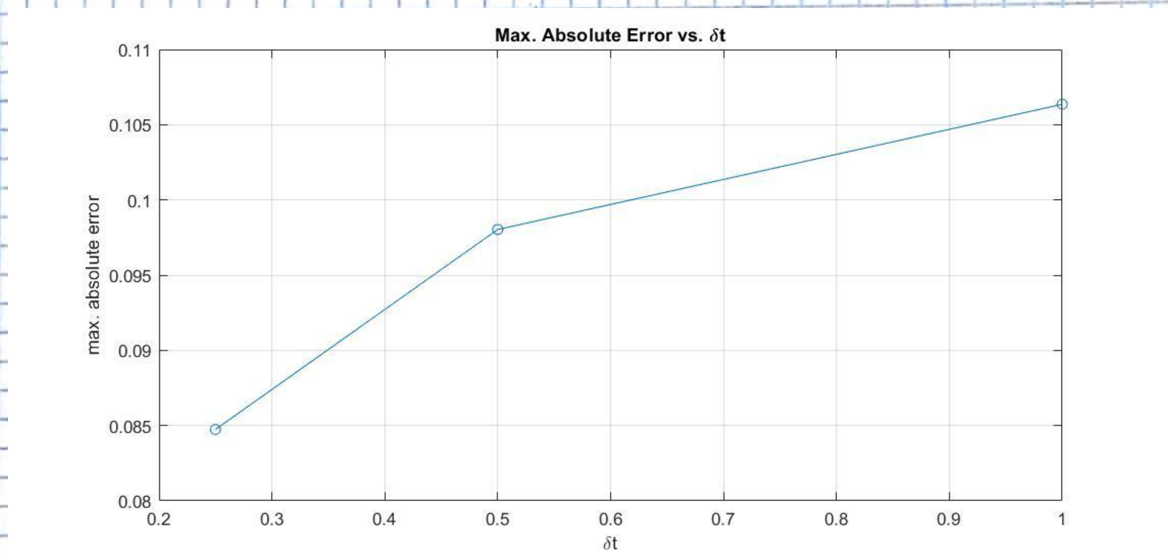
$$\left(\frac{1}{\Delta t} M + K\right) u^{n+1} = f + \frac{1}{\Delta t} M u_a^{n+1}$$

$$\left(\frac{1}{9\Delta t}\begin{bmatrix} 2 & 1/2 \\ 1/2 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\right)\begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \frac{1}{\Delta t} M \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \frac{6\Delta t}{54\Delta t + 5}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

3) Error of splitting approach with respect to monolithic approach.

Taking the maximum absolute difference between the solutions obtained using the monolithic and splitting approaches for a given Δt as our measure of error, we obtain the following plot for $\Delta t = [0{,}25 \quad 0{,}5 \quad 1]$



As we may see, the splitting approach technique converges to the monolithic solution as $\Delta t \to 0$. This is an expected behavior since the splitting approach introduces a splitting error of order $O(\Delta t)$.

HW 6

Fractional Step Methods

Yosida Scheme for incompressible Navier Stokes

(1) $\frac{1}{\Delta t} M(\hat{u}^{n+1} - u^n) + K\hat{u}^{n+1} = F - G\tilde{p}^{n+1}$

(2) $DM^{-1} G p^{n+1} = \frac{1}{\Delta t} D\hat{u}^{n+1} - DM^{-1} G\tilde{p}^{n+1}$

(3) $\frac{1}{\Delta t} M(u^{n+1} - \tilde{u}^{n+1}) + \alpha K(u^{n+1} - \tilde{u}^{n+1}) + G(p^{n+1} - \tilde{p}^{n+1}) = 0$

1.) Optimal value for α parameter

The monolithic form of the discretized Navier-Stokes eqs, using a BDF1 time discretization is given by

(4) $\frac{1}{\Delta t} M(u^{n+1} - u^n) + Ku^{n+1} + Gp^{n+1} = F$

(5) $Du^{n+1} = 0$

Adding equations (1) and (3) of the Yosida scheme yields:

$$\frac{1}{\Delta t} M(u^{n+1} - u^n) + K(\hat{u}^{n+1} + \alpha u^{n+1} - \alpha\hat{u}^{n+1}) + Gp^{n+1} = F$$

By setting $\alpha = 1$, we recover the original equation (4).
Therefore, we conclude that the optimal value is $\boxed{\alpha = 1}$

2.) Source of error of the scheme.

The approximation of the intermediate variables $\hat{u}$ and $\tilde{p}$ introduces a splitting error which results in a relaxation of the incompressibility condition.

HW 7

ALE Formulations

1.) Given the spatial description of a property

$$\gamma(x,y,z,t) = [2x,\ ye^t,\ z]$$

the eqs. of movement:

$$x = Xe^t \qquad y = Y + e^t - 1 \qquad z = Z$$

eqs. of movement of the mesh:

$$x_m = \chi + \alpha t \qquad y_m = \mathcal{Y} - \beta t \qquad z_m = \mathcal{Z}$$

a.) Description of the property in terms of ALE coordinates

$$\gamma_{ALE} = \gamma(\chi, \mathcal{Y}, \mathcal{Z}, t) = [2(\chi + \alpha t),\ (\mathcal{Y} - \beta t)e^t,\ \mathcal{Z}]$$

b.) Velocity of particles and mesh velocity.

- Particles

$$\frac{\partial x}{\partial t} = Xe^t \qquad \frac{\partial y}{\partial t} = e^t \qquad \frac{\partial z}{\partial t} = 0$$

$$\boxed{v = [Xe^t,\ e^t,\ 0]}$$

- Mesh

$$\frac{\partial x_m}{\partial t} = \alpha \qquad \frac{\partial y_m}{\partial t} = -\beta \qquad \frac{\partial z_m}{\partial t} = 0$$

$$\boxed{v_{mesh} = [\alpha,\ -\beta,\ 0]}$$

c) ALE description of material time derivative of r

$$\frac{d}{dt} r_{ALE} = \frac{\partial}{\partial t} r_{ALE} + (v - v_{mesh}) \cdot \nabla r(x,t)$$

$$\frac{\partial}{\partial t} r_{ALE} = \left[2\alpha \; , \; (y - \beta t - \beta)e^{t} \; , \; 0 \right]$$

$$\nabla r(x,t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v - v_{mesh} = \left[Xe^{t} - \alpha \; , \; e^{t} + \beta \; , \; 0 \right]$$

$$\frac{d}{dt} r_{ALE} = \frac{\partial}{\partial t} r_{ALE} + \left[2(Xe^{t} - \alpha) \; , \; e^{t}(e^{t} + \beta) \; , \; 0 \right]$$

$$\frac{d}{dt} r_{ALE} = \left[2Xe^{t} \; , (y - \beta t + e^{t})e^{t} \; , \; 0 \right]$$

Let us recall

$$X = xe^{-t} = (\mathcal{X} + \alpha t)e^{-t}$$

Replacing in the material derivative:

$$\boxed{\frac{d}{dt} r_{ALE} = \left[2(\mathcal{X} + \alpha t) \; , \; (y - \beta t + e^{t})e^{t} \; , \; 0 \right]}$$

2.) ALE form of incompressible Navier-Stokes eqs. Where (in time and space) is each of the terms evaluated? How are time derivatives computed?

- Momentum equation ALE formulation:

$$\frac{\partial}{\partial t} u_{ALE}(\chi,t) + c \cdot \nabla u(x,t) = \nabla \cdot \sigma(x,t) + \rho(x,t)\, b(x,t)$$

where $c = v - v_{mesh}$

Replacing the constitutive law for a Newtonian fluid

$$\sigma(x,t) = -p(x,t)\, I + 2\mu \nabla^s u(x,t)$$

into the momentum equation:

$$\boxed{\frac{\partial}{\partial t} u_{ALE}(\chi,t) + c \cdot \nabla u(x,t) - \mu \nabla^2 u(x,t) + \nabla p(x,t) = \rho(x,t)\, b(x,t)}$$

- Mass balance eq:

$$\underbrace{\cancel{\frac{\partial \rho_{ALE}}{\partial t}}}_{\text{incomp.}} + \cancel{c \cdot \nabla \rho(x,t)} = -\rho \nabla \cdot u(x,t) \longrightarrow \underbrace{\boxed{\nabla \cdot u(x,t) = 0}}_{\text{Remains unchanged}}$$

Regarding spatial discretization, all terms of the momentum equation are described in an Eulerian frame of reference, except for the time derivative which is described using ALE formulation. The mass balance equation of an incompressible flow is described in Eulerian formulation.

Regarding time discretization, the time at which each term is evaluated depends on the time integration scheme used.

3.) Bibliographical research on existing methods for the definition of mesh movement in ALE formulations. Describe the main advantages of each method.

Two types of techniques may be used for the definition of mesh movement in ALE formulations. The first of them consists of a mesh regularization (geometrical) approach, which aims to avoid mesh distortion and entanglement by keeping it as regular as possible during calculations. The second approach is mesh adaptation, which results in meshes with a concentration of elements in zones which require high precision due to steep solution gradients. Now we will shortly describe both approaches.

1 Mesh regularization techniques.

These are classified depending on whether boundary motion is prescribed or unknown. When boundary motion is known a priori, mesh motion is also prescribed a priori, generally using a Lagrangian description of boundary motion and Eulerian formulation for nodes far away from the moving boundaries, with a transition zone defined in between. Some examples of these techniques include:

- Transfinite mapping method:

  Originally designed to geometrically describe a mesh with specified boundaries. It defines an approximate surface or volume at a nondenumerable number of points. In 2D it can model all domain boundaries exactly and thus, no geometric error is introduced and it is also an inexpensive procedure. However, this method imposes restrictions on mesh topology.

- Laplacian smoothing:

  Also called elliptic mesh generation. It consists of solving a Laplace equation for each component of velocity or position in the mesh nodes. Its main disadvantage is that for a non-convex domain, nodes may run outside it.

- Mesh smoothing:

  Simple and general methods, relying on iterative algorithms to minimize the geometrical squeeze and distortion of mesh elements. These methods may be applied to unstructured meshes based in both triangular and quadrilateral elements

2. Mesh adaptation techniques:

   These methods consist in the use of ALE formulation as an adaptative technique, optimizing the mesh configuration to achieve improved accuracy and possibly lower computational cost since the number of elements and connectivity of the mesh remains unchanged througnout the computation. The ALE algorithm includes an indicator of error, and the mesh is modified to evenly distribute the error over the domain according to this indicator. This equidistribution can be carried out using elliptic or parabolic differential equations.

   This approach is widely used in solid mechanics simulations involving large deformations and in fluid dynamics in cases where the directional character of a flow must be taken into account.

Source: J. Donea, A. Huerta, J. Ph. Ponthot, A. Rodriguez-Ferran. Arbitrary Lagrangian-Eulerian Methods. Wiley, 2004

HW 8

Fluid-Structure Interaction

1.) Describe the added mass effect problem for fluid-structure interaction problems. When does it appear? What kind of problems suffer from it? What are the main methods for dealing with it?

The added mass effect is a numerical effect which keeps partitioned schemes from reaching convergence. It occurs when the fluid and the structure have similar densities. due to additional tractions imposed on the structure at the interface, coming from mass associated to inner nodes of the fluid domain.

Problems such as blood-tissue interaction in biomedical engineering are specially sensitive to this problem due to the similarity in the densities of the blood and the tissue. On the other hand, problems where the structure has a much higher density than the fluid, such as steel and air, partitioned schemes work well.

One of the most used solutions to this problem is the Aitken relaxation scheme, which uses a relaxation parameter that for each iteration, depends on the solution of the two previous iterations. Other options include steepest descent methods and Robin-Robin BC's.

2.) Iteration by subdomain scheme for heat transfer problem described in problem 4. Apply two iterations of Aitken scheme to it.

1D transient heat transfer:

$$\begin{cases} \dfrac{\partial u}{\partial t} - K \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } [0,1] \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Let us recall the discretized problem from page 19, which involves a FE space discretization and BDF1 time discretization:

$$\left(\frac{M}{\Delta t}+K\right)u^{n+1} = F + \frac{M}{\Delta t}u^n$$

where:

$$M = \int_\Omega N_i N_j \qquad K = \int_\Omega \nabla N_i \cdot K \nabla N_j \qquad F = \int_\Omega N_i f$$

Applying a Dirichlet - Neumann iteration by subdomain scheme:

- Ω_1 (Neumann problem)

$$A = \frac{M}{\Delta t} + K \qquad B^n = F + \frac{M}{\Delta t}u^n$$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{n+1,k} \\ u_{\Gamma *}^{n+1,k} \end{bmatrix} = \begin{bmatrix} B_1^n \\ B_\Gamma^n - A_{\Gamma 2}u_2^{n+1,k-1} - A_{\Gamma\Gamma}^{(2)}u_\Gamma^{n+1,k-1} \end{bmatrix}$$

$n \to$ time step

$k \to$ iteration

- Ω_2 (Dirichlet problem)

we must introduce the relaxation here.

$$u_\Gamma^{n+1,k} = \omega u_{\Gamma *}^{n+1,k} + (1-\omega)u_\Gamma^{n+1,k-1}$$

where ω is the relaxation parameter:

$$\omega = \frac{u_\Gamma^{n+1,k-2} - u_\Gamma^{n+1,k-1}}{u_\Gamma^{n+1,k-2} - u_\Gamma^{n+1,k-1} + u_{\Gamma *}^{n+1,k} - u_{\Gamma *}^{n+1,k-1}}$$

$\longrightarrow$ not valid for $\underline{k<2}$

First 2 iterations performed without relaxation.

$$\boxed{A_{22}u_2^{n+1,k} = B_2^n - A_{2\Gamma}u_\Gamma^{n+1,k}}$$

$\Downarrow$

Gauss-Seidel scheme

- we will perform the two Aitkens iterations starting from $\underline{K=2}$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1^{n+1,2} \\ U_{\Gamma *}^{n+1,2} \end{bmatrix} = \begin{bmatrix} B_1^n \\ B_\Gamma^n - A_{\Gamma 2} U_2^{n+1,1} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{n+1,1} \end{bmatrix} \Bigg\} \Omega_1 \text{ (Neumann)}$$

Solve for $U_{\Gamma *}^{n+1,2}$

$$\omega = \frac{U_\Gamma^{n+1,0} - U_\Gamma^{n+1,1}}{U_\Gamma^{n+1,0} - U_\Gamma^{n+1,1} + U_{\Gamma *}^{n+1,2} - U_{\Gamma *}^{n+1,1}}$$

$$U_\Gamma^{n+1,2} = \omega U_{\Gamma *}^{n+1,2} + (1-\omega) U_\Gamma^{n+1,1}$$

$$A_{22} U_2^{n+1,2} = B_2^n - A_{2\Gamma} U_\Gamma^{n+1,2} \longrightarrow \Omega_2 \text{ (Dirichlet)}$$

Solve for $U_2^{n+1,2}$

- update $K \longrightarrow \underline{K=3}$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1^{n+1,3} \\ U_{\Gamma *}^{n+1,3} \end{bmatrix} = \begin{bmatrix} B_1^n \\ B_\Gamma^n - A_{\Gamma 2} U_2^{n+1,2} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{n+1,2} \end{bmatrix} \Bigg\} \Omega_1$$

Solve for $U_{\Gamma *}^{n+1,3}$

$$\omega = \frac{U_\Gamma^{n+1,1} - U_\Gamma^{n+1,2}}{U_\Gamma^{n+1,1} - U_\Gamma^{n+1,2} + U_{\Gamma *}^{n+1,3} - U_{\Gamma *}^{n+1,2}}$$

$$U_\Gamma^{n+1,3} = \omega U_{\Gamma *}^{n+1,3} + (1-\omega) U_\Gamma^{n+1,2}$$

$$\boxed{A_{22} U_2^{n+1,3} = B_2^n - A_{2\Gamma} U_\Gamma^{n+1,3}} \quad \Omega_2$$

3.) Monolithic, transient (BDF1), finite element (linear, $h = 1/4$) approximation of the heat transfer eq. Enforce Dirichlet BC's at $x=0$ and $x=1$ using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

Linear shape functions $\rightarrow N_1 = \dfrac{x_2 - x}{h} \qquad N_2 = \dfrac{x - x_1}{h}$

$$\frac{dN_1}{dx} = -\frac{1}{h} \qquad \frac{dN_2}{dx} = \frac{1}{h} \qquad h = \frac{1}{4}$$

Let us recall the discrete form of the problem

$$\left(\frac{M}{\Delta t} + K\right) u^{n+1} = F + \frac{M}{\Delta t} u^n$$

$$M^{(e)} = \int_{\Omega_e} N_i N_j = h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$ (see page 25)

$$M = \frac{1}{24} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ & 4 & 1 & 0 & 0 \\ & & 4 & 1 & 0 \\ & & & 4 & 1 \\ \text{sym} & & & & 2 \end{bmatrix}$$

$$K^{(e)} = \int_{\Omega_e} \nabla N_i \, k \, \nabla N_j = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow K = k \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ & 8 & -4 & 0 & 0 \\ & & 8 & -4 & 0 \\ & & & 8 & -4 \\ \text{sym} & & & & 4 \end{bmatrix}$$

Lagrange multipliers are imposed as:

$$Hu = e$$

In order to apply the constraints $u(0) = 0$ and $u(1) = 0$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let us define:

$$A = \frac{M}{\Delta t} + K \qquad B^n = F + \frac{M}{\Delta t} u^n$$

$$\begin{bmatrix} A & H^T \\ H & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} B^n \\ e \end{bmatrix}$$

Assuming $\Delta t = 1$ and $\kappa = 1$ and replacing these values in the system, we obtain:

$$A = \frac{M}{\Delta t} + K = \frac{1}{24} \begin{bmatrix} 98 & -95 & 0 & 0 & 0 \\ & 196 & -95 & 0 & 0 \\ & & 196 & -95 & 0 \\ & & & 196 & -95 \\ \text{sym} & & & & 98 \end{bmatrix}$$

$$\begin{bmatrix} A & H^T \\ H & 0 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 98 & -95 & 0 & 0 & 0 & 24 & 0 \\ & 196 & -95 & 0 & 0 & 0 & 0 \\ & & 196 & -95 & 0 & 0 & 0 \\ & & & 196 & -95 & 0 & 0 \\ & & & & 98 & 0 & 24 \\ & & & & & 0 & 0 \\ \text{sym} & & & & & & 0 \end{bmatrix}$$

with the assumed values for Δt and κ, the condition number of the system is $\underline{\underline{38,3156}}$ (computed using Matlab).

4) Starting from the previous problem, suppose that a level set function ($\Psi = 0$ at $x = 0,4$) divides the domain into a high thermal conductivity ($\kappa = 100$) subdomain ($x \in [0, 0.4]$) and a low thermal conductivity ($\kappa = 1$) subdomain ($x \in (0.4, 1]$). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.

Only the stiffness matrix is affected:

$$K^{(e)} = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \text{For all elements } \underline{\text{but}} \text{ element } 2 \; (x \in [0{,}25 \, , 0{,}5])$$

$$\text{Element } 1 \longrightarrow K^{(1)} = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{Elements } 3 \text{ and } 4 \longrightarrow K^{(3)} = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = K^{(4)}$$

For the computation of $K^{(2)}$, we must perform a subintegration:

$$K^{(2)} = 100\int_{0{,}25}^{0{,}4} \nabla N_i \nabla N_j + \int_{0{,}4}^{0{,}5} \nabla N_i \nabla N_j$$

$$K^{(2)} = \frac{100}{h^2}(0{,}4-0{,}25)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{h^2}(0{,}5-0{,}4)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 16\left((15 + 0{,}1)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \longrightarrow K^{(2)} = 241{,}6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembling the global matrix:

$$K = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ & 641{,}6 & -241{,}6 & 0 & 0 \\ & & 245{,}6 & -4 & 0 \\ & & & 8 & -4 \\ \text{sym} & & & & 4 \end{bmatrix}$$

Assuming $\Delta t = 1$, and considering that M and H remain unchanged, the system matrix takes the form:

$$\begin{bmatrix} A & H^T \\ H & 0 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 9602 & -9599 & 0 & 0 & 0 & 24 & 0 \\ & 15402{,}4 & -5797{,}4 & 0 & 0 & 0 & 0 \\ & & 5898{,}4 & -95 & 0 & 0 & 0 \\ & & & 196 & -95 & 0 & 0 \\ & & & & 98 & 0 & 24 \\ & & & & & 0 & 0 \\ \text{sym} & & & & & & 0 \end{bmatrix}$$

$\boxed{\text{condition number: } 4696{,}8}$