Alba Navarro Casanova

PROBLEMS: COUPLED PROBLEMS

1. Transmission conditions

E1) $EI\dfrac{d^4v}{dx^4} = f$ $\xrightarrow{PVW}$ $EI\displaystyle\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f$

$v(0) = v(L) = 0$ clamped

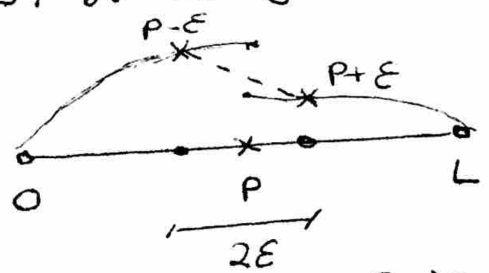
$\delta v(0) = \delta v(L) = 0$

$\dfrac{d\delta v}{dx}(0) = \dfrac{d\,\delta_v(L)}{dx} = L$

a) $v \in H^2$ $\quad H^2(\Omega) := \{ v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < \infty \mid \int_\Omega |\nabla v|^2 < \infty \mid \int_\Omega |\Delta v|^2 < \infty \}$. In order to satisfy $\dfrac{d^2v}{dx^2}$, the second derivative of "v" most remain bounded. Hence, "v" belongs to H^2.

$\delta v \in H_0^2 \quad H_0^2(\Omega) := \{ \delta v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < \infty \mid \int_\Omega |\nabla v|^2 < \infty \mid \int_\Omega |\Delta v|^2 < \infty \mid \delta v|_{\partial\Omega} = 0 \mid \frac{d\delta v}{dx} = 0 |_{\partial\Omega} \}$

As the previous case, "δv" must belong to H^2 for satisfying the term $\dfrac{d^2\delta v}{dx^2}$. Furthermore, "$\delta v$" and "$\dfrac{d\delta v}{dx}$" vanish on the boundaries. Thus, "δv" belongs to H_0^2, which is a subspace of H.

b) 

Assuming that $\left.\dfrac{d^2v}{dx^2}\right|_P = \lim_{\varepsilon\to 0} \left.\dfrac{d^2v^{\varepsilon}}{dx^2}\right|_P$

$$\int_0^{P-\varepsilon}\left(\frac{d^2v}{dx^2}\right)^2 + \underbrace{\int_{P-\varepsilon}^{P+\varepsilon}\left(\frac{d^2v}{dx^2}\right)^2}_{A} + \int_{P+\varepsilon}^{L}\left(\frac{d^2v}{dx^2}\right)^2 = \int_0^L\left(\frac{d^2v}{dx^2}\right)^2$$

If $v \in H^2$ this integral must remain bounded

$$A \quad \int_{P-\varepsilon}^{P+\varepsilon}\left(\frac{d^2v}{dx^2}\right)^2 \simeq (2\varepsilon)\underbrace{\left(\frac{v(\overset{\ominus}{P+\varepsilon}) - 2v(P) + v(\overset{\oplus}{P+\varepsilon})}{(2\varepsilon)^2}\right)^2}_{\text{second derivative approximation}} = \left[\frac{\left(v(\overset{\ominus}{P+\varepsilon}) - 2v(P) + v(\overset{\oplus}{P+\varepsilon})\right)^2}{(2\varepsilon)^3}\right]$$

$$\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 = \int_0^{P^- -\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 + \frac{(v(P^- + \varepsilon) - 2v(P) + v(P^+ + \varepsilon))^2}{(2\varepsilon)^3} + \int_{P^+ + \varepsilon}^{L} \left(\frac{d^2v}{dx^2}\right)^2$$

For $\varepsilon \to 0$ the term $\dfrac{(v(P^- + \varepsilon) - 2v(P) + v(P^+ + \varepsilon))^2}{(2\varepsilon)^3} \to \infty$, for avoid that $v(P^-) = v(P^+)$, hence $[\![v]\!]_P = 0$

First Transmission conditions : $[\![v]\!]_P = 0$

In the same way, $\displaystyle\int_{P-\varepsilon}^{P+\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2$ can be expressed as:

$$\int_{P-\varepsilon}^{P+\varepsilon} \left(\frac{d^2v}{dx^2}\right)^2 = \int_{P-\varepsilon}^{P+\varepsilon} \left[\frac{d}{dx}\left(\frac{dv}{dx}\right)\right]^2 \doteq (2\varepsilon)\left[\frac{d}{dx}\left(\frac{v(P^+ + \varepsilon) - v(P^- - \varepsilon)}{2\varepsilon}\right)\right]^2 =$$

$$= \frac{1}{2\varepsilon}\left[\frac{d}{dx}\left(v(P^+ + \varepsilon) - v(P^- - \varepsilon)\right)\right]^2$$

$$\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 = \int_0^{P^- -\varepsilon} \left[\frac{d}{dx}\left(\frac{dv}{dx}\right)\right]^2 + \frac{1}{2\varepsilon}\left[\frac{d}{dx}(v(P^+ + \varepsilon) - v(P^- + \varepsilon))\right]^2 +$$

$$+ \int_{P^+ + \varepsilon}^{L} \left[\frac{d}{dx}\left(\frac{dv}{dx}\right)\right]^2$$

For $\varepsilon \to 0$ the term $1/2\varepsilon \left(\frac{d}{dx}(v(P^+ + \varepsilon) - v(P^- + \varepsilon))\right)^2 \to \infty$, for avoid this :

$$\frac{dv(P^+)}{dx} = \frac{dv(P^-)}{dx}, \text{ hence } \left[\!\!\left[\frac{dv}{dx}\right]\!\!\right] = 0$$

Second transmission conditions $\left[\!\!\left[\dfrac{dv}{dx}\right]\!\!\right] = 0$

c) $EI \dfrac{d^4v}{dx^4} = f$

$$\int_0^L \delta v \, EI \frac{d^4v}{dx^4} = \int_0^L \delta v f \longrightarrow \underbrace{\int_0^{P^-} EI\,\delta v \frac{d^4v}{dx^4} + \int_{P^+}^{L} EI\,\delta v \frac{d^4v}{dx^4}}_{\text{to be integrated by parts}} =$$

$$= \int_0^{P^-} \delta v f + \int_{P^+}^{L} \delta v f$$

- $$\int_0^{P^-} EI\,\delta v \frac{d^4v}{dx^4} = EI\left[\delta v \frac{d^3v}{dx}\Big|_0^{P^-} - \int_0^{P^-} \frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]$$

- $\int_p^L EI\,\delta v \frac{d^4v}{dx^4} = EI\left[\delta v \frac{d^3v}{dx^3}\Big|_{p+}^{L} - \int_{p+}^{L} \frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]$

(*) $\delta v(0) = \delta v(L) = 0.$ $(\delta v \in H_0^2)$

$$EI\left[\delta v \frac{d^3v}{dx^3}\Big|_{\cancel{0}}^{p^-} + \delta v \frac{d^3v}{dx^3}\Big|_{p+}^{\cancel{L}} - \int_0^{p^-} \frac{d\delta v}{dx}\frac{d^3v}{dx^3} - \int_{p+}^{L} \frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right] =$$

$$= \int_0^{p^-} \delta v f + \int_{p+}^{L} \delta v f$$

Applying the additive property of the integral,

$$EI\left[\delta v \frac{d^3v}{dx^3}\Big|_{p-} - \delta v \frac{d^3v}{dx^3}\Big|_{p+} - \int_0^{L} \frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right] = \int_0^L \delta v f = 0$$

(for recover the original problem)

<u>Third Transmission condition</u>:

$$\delta v \frac{d^3v}{dx^3}\Big|_{p^-} = \delta v \frac{d^3v}{dx^3}\Big|_{p+}$$

$$-EI\int_0^L \frac{d\delta v}{dx}\frac{d^3v}{dx^3} = \int_0^L \delta v f$$

the integral can be splitted again,

$$-EI\underbrace{\left[\int_0^{p^-} \frac{d\delta v}{dx}\frac{d^3V}{dx^3} + \int_{p+}^{L} \frac{d\delta v}{dx}\frac{d^3v}{dx^3}\right]}_{\text{these terms can be integrated by parts}} = \int_0^{p^-} \delta v f + \int_{p+}^{L} \delta v f$$

So, after integrating by parts, yields:

$$-EI\left[\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{\cancel{0}\,(*)}^{p^-} + \frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{p+}^{\cancel{L}\,(*)} - \int_0^{p^-} \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2} - \int_{p+}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\right] =$$

$$= \int_0^{p^-} \delta v f + \int_{p+}^{L} \delta v f$$

(*) $\left(\frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0;\ \delta v \in H_0^2\right)$

Applying the additive property of the integral.

$$-EI\left[\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{p^-}-\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{p^+}\right]+EI\int_0^L\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}=\int_0^L\delta v f$$

$= 0$ (for recover the original problem)

4th Transmission condition $\quad \frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{p^-} = \frac{d\delta v}{dx}\frac{d^2v}{dx^2}\Big|_{p^+}$

E2)

$$\nu\nabla\times\nabla\times\underline{u} = f \quad \text{in } \Omega$$

$$\nabla\cdot\underline{u} = 0 \quad \text{in } \Omega$$

$$\underline{n}\times\underline{u} = 0 \quad \text{on } \partial\Omega$$

a) $\int_\Omega \underline{w}\cdot(\nu\nabla\times\nabla\times\underline{u}) = \int_\Omega \underline{w}\cdot f$

$$\nabla\cdot(\underline{w}\times(\nabla\times\underline{u})) = (\nabla\times\underline{w})\cdot(\nabla\times\underline{u}) + \underline{w}\cdot(\nabla\times\nabla\times\underline{u})$$

$$w\cdot(\nabla\times\nabla\times u) = \nabla\cdot(\underline{w}\times\nabla\times\underline{u}) - (\nabla\times w)\cdot(\nabla\times u)$$

$$\underbrace{\nu\int_\Omega\nabla\cdot(\underline{w}\times\nabla\times\underline{u})}_{\text{Gauss theorem}} - \nu\int_\Omega(\nabla\times\underline{w})\cdot(\nabla\times\underline{u}) = \int_\Omega\underline{w}\cdot f$$

Applying the Gauss theorem to the first term:

$$\nu\int_{\partial\Omega}(\underline{w}\times(\nabla\times\underline{u}))\cdot\underline{n} - \nu\int_\Omega(\nabla\times\underline{w})\cdot(\nabla\times u) = \int_\Omega\underline{w}\cdot f$$

By means of the cross identity: $a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$

$$\nu\int_{\partial\Omega}((\nabla\times\underline{u})\times\underline{n})\cdot\underline{w} - \nu\int_\Omega(\nabla\times\underline{w})\cdot(\nabla\times\underline{u}) = \int_\Omega\underline{w}\cdot f$$

Find $\underline{u}\in H_{curl}$ such that:

$$\nu\int_{\partial\Omega}((\nabla\times\underline{u})\times\underline{n})\cdot\underline{w} - \nu\int_\Omega(\nabla\times\underline{w})\cdot(\nabla\times\underline{u}) = \int_\Omega\underline{w}\cdot f$$

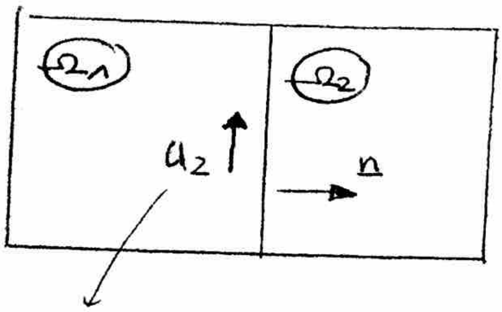
$\forall\,\underline{w}\in H_{curl}$.

$\underline{u}$ must belong to Hcurl, being Hcurl the space defined as:

$$H\text{curl}(\Omega) := \left\{ u : \Omega \to \mathbb{R}^3 \mid \int_\Omega |u^2| < \infty \mid \int_\Omega |\nabla \times u|^2 < \infty \right\}$$

b) The required regularity condition is $\int_\Omega |\nabla \times \underline{u}|^2 < \infty$

Taking as a sample a 2D domain:



Trial field

Considering a trial tangent field of the form: $(0, u_2(x,y))$ and computing its rotational:

$$\nabla \times (0, u_2(x,y)) = \begin{vmatrix} \underline{x} & \underline{y} & \underline{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & u_2 & 0 \end{vmatrix} =$$

$$= \frac{\partial u_2}{\partial x} \underline{z} - \cancel{\frac{\partial u_2}{\partial z}}^{=0} \underline{x}$$

As can be seen, the tangent field fulfils: $\int_\Omega \left| \frac{\partial u_2}{\partial x} \right|^2 < \infty$

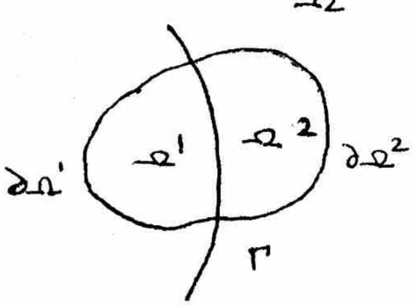
As a consequence, since $u_{tg} = \underline{n} \times \underline{\;}$

$$\boxed{[\![\underline{n} \times \underline{u}]\!] = 0}$$

c) Transmission conditions from the additive imposition of the integral

Recovering the variational form: $v \int_{\partial\Omega} ((\nabla \times \underline{u}) \times \underline{n}) \cdot w - v \int_\Omega (\nabla \times \underline{w}) \cdot$

• $(\nabla \times \underline{u}) = \int_\Omega \underline{w} \cdot f$



If the integral is additive, it can be splitted by subdomains as follows:

$$v \int_{\partial\Omega_1} ((\nabla \times \underline{u}) \times \underline{n}_1) \cdot \underline{w} + v \int_{\partial\Omega_2} ((\nabla \times \underline{u}) \times \underline{n}_2) \cdot w - v \int_{\Omega_1} (\nabla \times \underline{w}) \cdot (\nabla \times u) - v \int_{\Omega_2} ($$

$$\partial\Omega_1 = \Gamma \cup (\partial\Omega \cap \Omega_1)^{0} \qquad \partial\Omega_2 = \Gamma \cup (\partial\Omega \cap \Omega_2)^{0}$$

$$w = 0 \text{ on } \partial\Omega^{*} \qquad w = 0 \text{ on } \partial\Omega^{*}$$

$$(\nabla \times \underline{w}) \cdot (\nabla \times u) = \int_{\Omega_1} \underline{w} \cdot \underline{f} + \int_{\Omega_2} \underline{w} \cdot \underline{f}$$

* selecting test functions that vanish at the Dirichlet boundary

Since the integral is additive : $\Omega = \Omega_1 \cup \Omega_2$

$$\underbrace{\nu \int_{\Gamma} ((\nabla \times \underline{u}) \times \underline{n}_1) \cdot \underline{w} + \nu \int_{\Gamma} ((\nabla \times \underline{u}) \times \underline{n}_2) \cdot \underline{w}} - \nu \int_{\Omega} (\nabla \times \underline{w}) \cdot (\nabla \times \underline{u}) =$$

$$= \int_{\Omega} \underline{w} \cdot \underline{f}$$

These terms must vanish for recover the original form (considering that $w = 0$ on ∂.)

$$\nu \int_{\Gamma} ((\nabla \times \underline{u}) \times \underline{n}_1) \cdot \underline{w} + \nu \int_{\Gamma} ((\nabla \times \underline{u}) \times \underline{n}_2) \cdot w = 0$$

$$\boxed{\nu \int_{\Gamma} [(\nabla \times \underline{u}) \times \underline{n}_1 + (\nabla \times \underline{u}) \times \underline{n}_2] \cdot w = 0 \quad \forall w \in H_{curl} \Rightarrow \text{Second Transmission Condition!}}$$

E3) Nawer equations for elastic material:

a) Variational form:

a.1)
$$\left.\begin{aligned} -2\mu \nabla \cdot (\underline{\underline{\varepsilon}}(\underline{u})) - \lambda \nabla (\nabla \cdot \underline{u}) &= \rho \underline{b} \quad \text{in } \Omega \\ \underline{u} &= \underline{0} \quad \text{on } \partial\Omega \end{aligned}\right\}$$

$$\underbrace{- \int_{\Omega} \underline{w} \cdot (2\mu \nabla \cdot (\underline{\underline{\varepsilon}}(\underline{u}))}_{A} - \underbrace{\int_{\Omega} \underline{w} \cdot (\lambda \nabla (\nabla \cdot \underline{u}))}_{B} = \int_{\Omega} \underline{w} \cdot (\rho \underline{b})$$

A) $$- \int_{\Omega} \underline{w} (2\mu \nabla \cdot (\underline{\underline{\varepsilon}}(\underline{u})) = -2\mu \int_{\Omega} \underbrace{(\nabla \cdot (\underline{w} \cdot \underline{\underline{\varepsilon}}(\underline{u})))}_{\text{Divergence theorem:}} - \nabla \underline{w} : \underline{\underline{\varepsilon}}(\underline{u})) =$$

$$\int_{\Omega} (\nabla \cdot (\underline{w} \cdot \underline{\underline{\varepsilon}}(\underline{u})) = \int_{\partial\Omega} (\underline{w} \cdot \underline{\underline{\varepsilon}}(\underline{u})) \cdot n =$$

0 $\underline{w}$ is chosen st $\underline{w} = 0$ on Γ_D

$$= 2\mu \int_{\Omega} \nabla \underline{w} : \underline{\underline{\varepsilon}}(\underline{u}) - 2\mu \int_{\partial\Omega} (\cancel{\underline{w}} \cdot \underline{\underline{\varepsilon}}(\underline{u})) \cdot \underline{n}$$

B) $$- \int_{\Omega} \underline{w} \cdot (\lambda \nabla (\nabla \cdot \underline{u})) = -\lambda \int_{\Omega} \underbrace{(\nabla \cdot (\underline{w} (\nabla \cdot \underline{u}))}_{\text{Divergence theorem:}} - (\nabla \cdot \underline{w})(\nabla \cdot u)) =$$

$$= \lambda \int_{\Omega} (\nabla \cdot \underline{w}) (\nabla \cdot \underline{u}) - \lambda \int_{\partial\Omega} (\underline{w} (\cancel{\nabla} \cdot \underline{u})) \cdot n$$

0 $\underline{w}$ is chosen st $w = 0$ on Γ_D.

Variational form of the problem:

Find $u \in H_0^1$ such that:

$$2\mu \int_\Omega \underline{\nabla}\,\underline{w} : \underline{\underline{\varepsilon}}(\underline{u}) + \lambda \int_\Omega (\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{w}\cdot(\rho b) \quad \forall w \in H_0^1$$

It has been considered that the whole boundary has Dirichlet boundary conditions.

a.2) $$\left.\begin{aligned} -\mu\underline{\Delta}\underline{u} - (\lambda+\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) &= \rho\underline{b} \quad \text{in } \Omega \\ \underline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}$$

$$\underbrace{-\int_\Omega \underline{w}\cdot(\mu\underline{\Delta}\underline{u})}_{A} - \underbrace{\int_\Omega \underline{w}\cdot((\lambda+\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}))}_{B} = \int_\Omega \underline{w}\cdot(\rho b)$$

A) $$-\int_\Omega \underline{w}\cdot(\mu\underline{\Delta}\underline{u}) = -\mu\int_\Omega \underline{w}\,\underline{\nabla}\cdot(\underline{\nabla}\underline{u}) = \underbrace{-\mu\int_\Omega \underline{\nabla}\cdot(\underline{w}\cdot(\underline{\nabla}\underline{u}))}_{\text{Divergence theorem}} - \underline{\nabla}\underline{w}:\nabla u =$$

$$= -\mu\int_{\partial\Omega}(\overset{0 \;\; w=0 \text{ on } \Gamma_0}{\cancel{\underline{w}}}\cdot(\underline{\nabla}\underline{u}))\cdot n + \mu\int_\Omega \underline{\nabla}\underline{w}:\underline{\nabla}\underline{u}$$

B) Is the same term that in the previous equation with different coefficients.

$$-(\lambda+\mu)\int_\Omega \underline{w}\cdot(\underline{\nabla}(\underline{\nabla}\cdot u)) = (\lambda+\mu)\left[\int_\Omega(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) - \underbrace{\int_{\partial\Omega}(\cancel{\underline{w}}\cdot(\underline{\nabla}\cdot\underline{u}))\cdot n}_{=0 \;\; w=0 \text{ on } \Gamma_0}\right]$$

Find $u \in H_0^1$ such that

$$\mu\int_\Omega \underline{\nabla}\underline{w} : \underline{\nabla}\underline{u} + (\lambda+\mu)\int_\Omega(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{w}\cdot(\rho b) \quad \forall w \in H_0^1$$

It has been considered that the whole boundary has Dirichlet BC's

a.3) $$\left.\begin{aligned} \mu\underline{\nabla}\times(\underline{\nabla}\times\underline{u}) - (\lambda+2\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) &= \rho\underline{b} \quad \text{in } \Omega \\ \underline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}$$

$$\underbrace{\int_\Omega \underline{w}\,(\mu\underline{\nabla}\times(\underline{\nabla}\times\underline{u}))}_{A} - \underbrace{\int_\Omega \underline{w}\,((\lambda+2\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}))}_{B} = \int_\Omega \underline{w}\cdot(\rho b)$$

A) $\int_\Omega \underline{w}\cdot(\mu \underline{\nabla}\times\underline{\nabla}\times\underline{u}) = \mu \underbrace{\int_\Omega \underline{\nabla}\cdot(\underline{w}\times(\underline{\nabla}\times u))}_{\text{Divergence theorem}} - (\underline{\nabla}\times\underline{w})\cdot(\nabla\times u) =$

$= \mu \int_{\partial\Omega} (\overset{0;\ w=0 \text{ on } \Gamma_D}{\cancel{\underline{w}}}\times(\underline{\nabla}\times\underline{u}))\cdot\underline{n} - \mu\int_\Omega (\underline{\nabla}\times\underline{w})\cdot(\underline{\nabla}\times\underline{u})$

B) Is the same as previous:

$-(\lambda+2\mu)\int_\Omega w\cdot(\underline{\nabla}(\underline{\nabla}\cdot\underline{u})) = \int_{\partial\Omega} \overset{0\quad w=0 \text{ on } \Gamma_D}{\cancel{\underline{w}}}(\underline{\nabla}\cdot\underline{u})\cdot\underline{n} - \int_\Omega(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u})$

Find $u\in H_0^1$ such that

$$-\mu\int_\Omega(\underline{\nabla}\times\underline{w})\cdot(\underline{\nabla}\times\underline{u}) + (\lambda+2\mu)\int_\Omega(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) = \int_\Omega \underline{w}\cdot(\rho b)$$

$\forall w\in H_0^1$. It has been considered that the whole boundary has Dirichlet boundary conditions.

b) obtain the transmission conditions by imposing that the integral is additive.

a) $-2\mu\underline{\nabla}\cdot(\underline{\underline{\varepsilon}}(\underline{u})) - \lambda\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = \rho\underline{b}$

Recovering the weak form:

$2\mu\int_{\Omega_1}\underline{\nabla}\,\underline{w} : \underline{\underline{\varepsilon}}(u) + 2\mu\int_{\Omega_2}\underline{\nabla}\,\underline{w} : \underline{\underline{\varepsilon}}(\underline{u}) - 2\mu\int_{\partial\Omega=\cancel{\Gamma_D^1}\cup\Gamma}(\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot n_1 -$

$-2\mu\int_{\partial\Omega=\cancel{\Gamma_D^2}\cup\Gamma}(\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot n_2 + \lambda\int_{\Omega_1}(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) + \lambda\int_{\Omega_2}(\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) -$

$-\lambda\int_{\partial\Omega=\cancel{\Gamma_D}\cup\Gamma}(\underline{w}(\underline{\nabla}\cdot u))\cdot n_1 - \lambda\int_{\partial\Omega=\cancel{\Gamma_D}\cup\Gamma}(\underline{w}(\underline{\nabla}\cdot\underline{u}))\cdot n_2 = \int_{\Omega_1}\underline{w}\cdot\rho\underline{b} + \int_{\Omega_2}\underline{w}\,\rho\underline{b}$

It has been considered that there is no Neumann BC and that w is chosen such that $w=0$ on Γ_D

Imposing that the integral is additive:

$$2\mu\int_\Omega \nabla \underline{w} : \underline{\underline{\varepsilon}}(\underline{u}) + \lambda\int_\Omega (\nabla\cdot\underline{w})(\nabla\cdot\underline{u}) \underbrace{- 2\mu\int_\Gamma (\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot\underline{n}_1 - 2\mu\int_\Gamma(\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot\underline{n}_2 - \lambda\int_\Gamma(\underline{w}(\nabla\cdot\underline{u}))\cdot\underline{n}_1 - \lambda\int_\Gamma(\underline{w}(\nabla\cdot\underline{u}))\cdot\underline{n}_2}_{*} =$$

$$= \int_\Omega \underline{w}\cdot(\rho\underline{b})$$

* For recovering the original form these terms must vanish!

considering $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$:

$$\boxed{\begin{aligned}&\int_\Gamma (\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot\underline{n}_1 + (\underline{w}\cdot\underline{\underline{\varepsilon}}(\underline{u}))\cdot\underline{n}_2 = 0\\ &\int_\Gamma (\underline{w}(\nabla\cdot\underline{u}))\cdot n_1 + (\underline{w}(\nabla\cdot\underline{u}))\cdot n_2 = 0\end{aligned}}$$

b) $-\mu\Delta\underline{u} - (\lambda+\mu)\nabla(\nabla\cdot\underline{u}) = \rho\underline{b}$

Recovering the weak form:

$$\mu\int_{\Omega_1}\nabla\underline{w} : \nabla\underline{u} + \mu\int_{\Omega_2}\nabla\underline{w} : \nabla\underline{u} - \int_{\partial\Omega = \cancel{\Gamma_0}\cup\Gamma} (w\cdot(\nabla\cdot\underline{u}))\cdot n_1 - \int_{\partial\Omega = \cancel{\Gamma_0}\cup\Gamma}(w\cdot(\nabla\underline{u}))\cdot$$

$$\cdot n_2 + (\lambda+\mu)\int_\Omega(\nabla\cdot\underline{w})(\nabla\cdot\underline{u}) + (\lambda+\mu)\int_{\Omega_2}(\nabla\cdot\underline{w})(\nabla\cdot\underline{u}) - (\lambda+\mu)\int_{\partial\Omega = \cancel{\Gamma_0}\cup\Gamma}(\underline{w}(\nabla\cdot\underline{u}))\cdot$$

$$\cdot n_1 - (\lambda+\mu)\int_{\partial\Omega = \cancel{\Gamma_0}\cup\Gamma}(\underline{w}\cdot(\nabla\cdot\underline{u}))\cdot n_2 = \int_{\Omega_1}\underline{w}\cdot(\rho\underline{b}) + \int_{\Omega_2}\underline{w}\cdot(\rho\underline{b})$$

The same Dirichlet BC assumption has been taken imposing the additive property of the integral.

$$\mu\int_\Omega\nabla\underline{w} : \nabla\underline{u} \underbrace{-\int_\Gamma(\underline{w}\cdot(\nabla\underline{u}))\cdot n_1 - \int_\Gamma(w\cdot(\nabla\underline{u}))\cdot n_2}_{\circledast} + (\lambda+\mu)\int_\Omega(\nabla\cdot\underline{w})$$

$$(\nabla\cdot u) \underbrace{- (\lambda+\mu)\int_\Gamma(\underline{w}\cdot(\nabla\cdot u))\cdot n_1 - (\lambda+\mu)\int_\Gamma(\underline{w}\cdot(\nabla\cdot u))\cdot n_2}_{\circledast} =$$

$$= \int_\Omega \underline{w}\cdot(\rho\underline{b})$$

* Terms that vanish for recovering the original form. Considering

$\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$

$$\boxed{\begin{aligned} &\int_\Gamma (\underline{w} \cdot (\underline{\nabla}\,\underline{u})) \cdot n_1 + (\underline{w} \cdot (\underline{\nabla}\,\underline{u})) \cdot n_2 = 0 \\ &\int_\Gamma (\underline{w}\ (\underline{\nabla}\cdot\underline{u})) \cdot n_1 + (w(\nabla\cdot u)) \cdot n_2 = 0 \end{aligned}}$$

c) $\mu \underline{\nabla} \times (\underline{\nabla} \times \underline{u}) - (\lambda + 2\mu) \underline{\nabla}(\underline{\nabla}\cdot\underline{u}) = 0$

Recovering the weak form:

$$-\mu \int_{\Omega_1} (\nabla \times w) \cdot (\nabla \times \underline{u}) - \mu \int_{\Omega_2} (\nabla \times w) \cdot (\nabla \times \underline{u}) + \mu \int_{\partial\Omega = \cancel{\Gamma_0} \cup \Gamma} (\underline{w} \times (\nabla \times \underline{u})) : n_1 +$$

$$+ \mu \int_{\partial\Omega = \cancel{\Gamma_0} \cup \Gamma} (\underline{w} \times (\underline{\nabla} \times \underline{u})) \cdot \underline{n}_2 + (\lambda + 2\mu) \int_{\Omega_1} (\underline{\nabla}\cdot\underline{w})(\nabla\cdot\underline{u}) + (\lambda + 2\mu) \int_{\Omega_2} (\underline{\nabla}\cdot\underline{w})$$

$$(\underline{\nabla}\cdot\underline{u}) - (\lambda + 2\mu) \int_{\partial\Omega = \cancel{\Gamma_0} \cup \Gamma} \underline{w}\ (\underline{\nabla}\cdot\underline{u}) \cdot \underline{n}_1 - (\lambda + 2\mu) \int_{\partial\Omega = \cancel{\Gamma_0} \cup \Gamma} \underline{w}\ (\underline{\nabla}\cdot\underline{u}) \cdot \underline{n}_2 =$$

$$= \int_{\Omega_1} \underline{w} \cdot (\rho \underline{b}) + \int_{\Omega_2} \underline{w} \cdot \rho \underline{b})$$

we have take the same :chlet BC assumption. Imposing the additive property of the integral:

$$-\mu \int_\Omega (\underline{\nabla} \times \underline{w}) \cdot (\underline{\nabla} \times \underline{u}) + \underbrace{\mu \int_\Gamma (\underline{w} \times (\underline{\nabla} \times \underline{u})) \cdot \underline{n}_1 + \mu \int_\Gamma (w \times (\underline{\nabla} \times \underline{u})) \cdot n_2}_{\circledast} +$$

$$+ (\lambda + 2\mu) \int_\Omega (\underline{\nabla}\cdot\underline{w})(\underline{\nabla}\cdot\underline{u}) - \underbrace{(\lambda + 2\mu) \int_\Gamma \underline{w}\ (\underline{\nabla}\cdot\underline{u}) n_1 - (\lambda + 2\mu) \cdot}_{\circledast}$$

$$\underbrace{\cdot \int_\Gamma \underline{w}\ (\underline{\nabla}\cdot\underline{u}) \cdot \underline{n}_2}_{\circledast} = \int_\Omega w \cdot (\rho \underline{b})$$

⊛ Terms that vanish for recovering the original form!

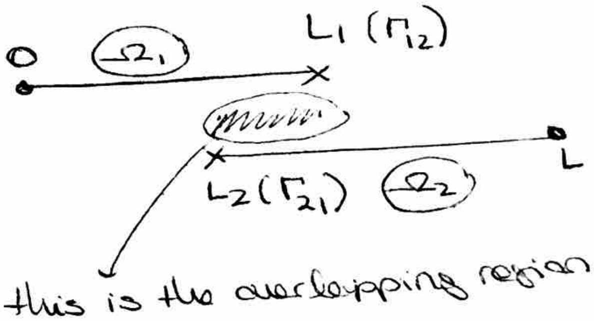
Considering that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$

$$\boxed{\begin{aligned} &\int_\Gamma (\underline{w} \times (\nabla \times \underline{u}))\cdot \underline{n}_1 + (\underline{w} \times (\nabla \times \underline{u}))\cdot \underline{n}_2 = 0 \\ &\int_\Gamma \underline{w} (\nabla \cdot \underline{u})\cdot \underline{n}_1 + \underline{w} (\nabla \cdot \underline{u})\cdot \underline{n}_2 = 0 \end{aligned}}$$

2. Domain decomposition methods

E1) Exercise 1.

$$\left.\begin{aligned} &EI \frac{d^4 v}{dx^4} = f \\ &v(0) = v(L) = 0 \\ &\frac{dv(0)}{dx} = \frac{dv(L)}{dx} = 0 \end{aligned}\right\}$$



a) IBS scheme based on a Schwartz additive DDM.

Subdomain 1:

$$EI \frac{d^4 v_1^{(k)}}{dx} = f \quad \text{in } \Omega_1$$

$$v_1^{(k)} = 0 \quad \text{on } x = 0$$

$$\frac{dv_1^{(k)}}{dx} = 0 \quad \text{on } x = 0$$

$$v_1^{(k)} = v_2^{(k-1)} \;(*) \quad \text{on } x = L_1$$

$$\frac{dv_1^{(k)}}{dx} = \frac{dv_2^{(k-1)}}{dx} \;(*) \quad \text{on } x = L_1$$

Subdomain 2:

$$EI \frac{d^4 v_2^{(k)}}{dx} = f \quad \text{in } \Omega_2$$

$$v_2^{(k)} = 0 \quad \text{on } x = L$$

$$\frac{dv_2^{(k)}}{dx} = 0 \quad \text{on } x = L$$

$$v_2^{(k)} = v_1^{(k-1)} \;(*) \quad \text{on } x = L_2$$

$$\frac{dv_2^{(k)}}{dx} = \frac{dv_1^{(k-1)}}{dx} \quad \text{on } x = L_2$$

(*) → Additive method: Transmission conditions obtained from the previous iteration.

b) FE space discretization: Matrix version

Recovering the weak form: $EI \int_\Omega \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} = \int_\Omega \delta v f$

It's required to interpolate both v and $\frac{dv}{dx}$ → so 2-noded Euler-Bernoulli beam element.

2-noded Euler-Bernoulli beam element: $\begin{bmatrix} v_1 \\ \frac{dv_1}{dx} \end{bmatrix}$ ① ——— ② $\begin{bmatrix} v_2 \\ \frac{dv_2}{dx} \end{bmatrix}$

$$\underline{\underline{K}}^{(e)} = \begin{bmatrix} 12 & 6\ell^{(e)} & -12 & 6\ell^{(e)} \\ & 4(\ell^{(e)})^2 & -6\ell^{(e)} & 2(\ell^{(e)})^2 \\ & & 12 & -6(\ell^{(e)}) \\ \text{symm.} & & & 4(\ell^{(e)})^2 \end{bmatrix} \left(\frac{EI}{(\ell^{(e)})^3}\right)$$

$$\underline{f}^{(e)} = \begin{bmatrix} 1/2 \\ \ell^{(e)}/12 \\ 1/2 \\ -\ell^{(e)}/12 \end{bmatrix} \qquad \underline{a}^{(e)} = \begin{bmatrix} v_1 \\ dv_1/dx \\ v_2 \\ dv_2/dx \end{bmatrix}$$

, the global system is:

$$\underline{\underline{K}}\ \underline{a} = \underline{f}$$

Matrix version of the previously stated IBS scheme:

Subdomain 1: $\underline{\underline{K}}_1 \underline{v}_1^{(k)} = \underline{f}_1$

$v_1^{(k)}(0) = 0$

$\frac{dv_1^{(k)}(0)}{dx} = 0$

$v_1^{(k)}(L_1) = v_2^{(k-1)}(L_1)$

$\frac{dv_1^{(k)}(L_1)}{dx} = \frac{dv_2^{(k-1)}(L_1)}{dx}$

Subdomain 2: $\underline{\underline{K}}_2 \underline{v}_2^{(k)} = \underline{f}$

$v_2^{(k)}(L) = 0$

$\frac{dv_2^{(k)}(L)}{dx} = 0$

$v_2^{(k)}(L_2) = v_1^{(k-1)}(L_2)$

$\frac{dv_2^{(k)}(L_2)}{dx} = \frac{dv_1^{(k-1)}(L_2)}{dx}$

E2) Exercise 2.

$u : \Omega \to \mathbb{R}^3$ such that

$$\nu \nabla \times \nabla \times \underline{u} = \underline{f} \text{ in } \Omega$$
$$\nabla \cdot \underline{u} = 0 \text{ in } \Omega$$
$$\underline{n} \times u = 0 \text{ on } \partial\Omega$$

a) IBS scheme based on D-N coupling. Recovering the previously obtained weak form:

$$\nu \int_{\partial\Omega} (\underline{w} \times (\nabla \times \underline{u})) \cdot \underline{n} - \nu \int_{\Omega} (\nabla \times \underline{w}) \cdot (\nabla \times \underline{u}) = \int_{\Omega} \underline{w} \cdot \underline{f}$$

and transmission conditions:

- $[\![n \times u]\!] = 0$
- $\int_\Gamma (\underline{w} \times (\nabla \times \underline{u})) \cdot n_1 = - \int_\Gamma (\underline{w} \times (\nabla \times \underline{u})) \cdot n_2$

Subdomain 1 (Dirichlet)

$$\begin{cases} \nu \int_{\partial\Omega_1} (\underline{w} \times (\nabla \times \underline{u}_1)) \cdot n_1 - \nu \int_{\Omega_1} (\nabla \times \underline{w}) \cdot (\nabla \times \underline{u}_1^{(k)}) = \int_{\Omega_1} \underline{w} \cdot f \quad \text{in } \Omega_1 \\ \underline{n}_1 \times \underline{u}_1^{(k)} = 0 \quad \text{on } \partial\Omega \cap \Omega_1 \\ n_1^{(k)} \times u_1^{(k)} = n_2 \times u_2^{(k-1)} \quad \text{on } \Gamma \end{cases}$$

Subdomain 2 (Neumann)

$$\begin{cases} \nu \int_{\partial\Omega_2} (\underline{w} \times (\nabla \times \underline{u}_2^{(k)})) \cdot \underline{n}_2 - \nu \int_{\Omega_2} (\nabla \times \underline{w}) \cdot (\nabla \times \underline{u}_2^{(k)}) = \int_{\Omega_2} \underline{w} \cdot f \quad \text{in } \Omega_2 \\ \underline{n}_2 \times \underline{u}_2^{(k)} = 0 \quad \text{on } \partial\Omega \cap \Omega_2 \\ \int_\Gamma (\underline{w} \times (\nabla \times \underline{u}_2^{(k)})) \cdot \underline{n}_2 = \int_\Gamma (\underline{w} \times (\nabla \times \underline{u}_1^{(\ell)})) \cdot n_1 \quad \text{on } \Gamma \end{cases}$$

(ℓ) $\begin{cases} \ell = k-1 \rightarrow \text{Jacobi Scheme} \\ \ell = k \rightarrow \text{Gauss-Seidel Scheme} \end{cases}$

b) obtain the expression of the Steklov-Poincaré operator:

$u_i = u_i^0 + \hat{u}_i \qquad i = 1,2$

$(\mathcal{L} = \nu \nabla \times \nabla \times)$

$$\left\{ \begin{array}{l} \mathcal{L} u_i^0 = f \quad \text{in } \Omega_i \\ n \times u_i^0 = 0 \quad \text{on } \partial\Omega \cap \Omega_i \\ u_i^0 = 0 \quad \text{on } \Gamma \end{array} \right\} \qquad \begin{array}{l} \mathcal{L} \hat{u}_i = 0 \quad \text{in } \Omega_i \\ n \times \hat{u}_i = 0 \quad \text{on } \partial\Omega \cap \Omega_i \\ \hat{u}_i = \varphi \quad \text{on } \Gamma \end{array}$$

Second transmission condition: $(\underline{w} \times \nabla \times \underline{u}_1)) \cdot n_1 = (\underline{w} \times \nabla \times \underline{u}_2) \cdot \underline{n}_2$

If we substitute the splitted solution in the second trans. cond yields

$$(\underline{w} \times (\nabla \times (\underline{u}_1^0 + \tilde{\underline{u}}_1))) \cdot n_1 = (\underline{w} \times (\nabla \times (\underline{u}_2^0 + \tilde{u}_2))) \cdot n_2$$

$$(\underline{w} \times (\nabla \times u_1^0)) \cdot n_1 + (\underline{w} \times (\nabla \times \tilde{\underline{u}}_1)) \cdot n_1 = (\underline{w} \times (\nabla \times u_2^0)) \cdot n_2 +$$

$$+ (\underline{w} \times (\nabla \times \tilde{\underline{U}}_2)) \cdot n_2$$

$$(\underline{w} \times (\nabla \times u_1^0)) \cdot n_1 - (\underline{w} \times (\nabla \times U_2^0)) \cdot n_2 = (\underline{w} \times (\nabla \times \tilde{\underline{U}}_2)) \cdot n_2 -$$

$$- (\underline{w} \times (\nabla \times \tilde{\underline{U}}_1)) \cdot n_1$$

$$\mathcal{L} : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$$

$$\varphi \longrightarrow (\underline{w} \times (\nabla \times \tilde{\underline{U}}_2)) \cdot n_2 - (\underline{w} \times (\nabla \times \tilde{\underline{U}}_1)) \cdot n_1$$

$$\mathcal{G} = (\underline{w} \times (\nabla \times u_1^0)) \cdot \underline{n}_1 - (\underline{w} \times (\nabla \times \underline{u}_2^0)) \cdot \underline{n}_2 \in H^{-1/2}(\Gamma)$$

Find $\varphi \in H^{1/2}(\Gamma)$ such that $\mathcal{L}\varphi = \mathcal{G}$

c) Matrix version of the problem, FE discret.

$$\underline{w} = N_i$$

$$\underline{u} \doteq \underline{u}^h = \sum_j N_j u_j$$

Weak form: $\underbrace{\nu \int_{\partial\Omega} (\underline{w} \times (\nabla \times \underline{u})) \cdot n}_{\underline{\underline{B}}} - \underbrace{\nu \int_{\Omega} (\nabla \times \underline{w}) \cdot (\nabla \times \underline{u})}_{\underline{\underline{K}}} = \underbrace{\int_{\Omega} \underline{w}\, f}_{\underline{f}}$

$$B_{ij} = \int_{\Gamma(e)} N_i \times (\nabla \times N_j) \cdot n$$

$$K_{ij} = \int_{\Omega(e)} (\nabla \times N_i) \cdot (\nabla \times N_j)$$

$$f_i = \int_{\Omega(e)} N_i f_i$$

Problem in Ω_1

$$\begin{cases} \nabla(\underline{\underline{B}}_1 - \underline{\underline{K}}_1)\, U_1^{(k)} = f & \text{in } \Omega_1 \\ \Pi \times U_1^{(k)} = 0 & \text{on } \partial\Omega \cap \Omega_1 \\ \underline{\underline{H}}_1 \times \underline{U}_1^{(k)} = \underline{\underline{H}}_2 \times \underline{U}_2^{(k-1)} & \text{on } \Gamma \end{cases}$$

Problem in Ω_2:

$$\begin{cases} \nabla(\underline{\underline{B}}_2 - \underline{\underline{K}}_2)\, \underline{U}_2^{(k)} = f & \text{in } \Omega_2 \\ \Pi \times U_2^{(k)} = 0 & \text{on } \partial\Omega \cap \Omega_2 \\ \underline{\underline{B}}_2\, \underline{U}_2^{(k)} = -\underline{\underline{B}}_1\, \underline{U}_1^{(e)} & \text{on } \Gamma \end{cases}$$

E3) Exercise 3:

$u : \Omega \longrightarrow \mathbb{R}$ such that

$$-k\Delta u = f \text{ in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$k > 0$, let Γ be a surface crossing Ω.

a) IBS based on Dirichlet - Robin coupling.

Subdomain 1

$$\begin{cases} -k_1 \Delta u_1^{(k)} = f & \text{in } \Omega_1 \\ U_1^{(k)} = 0 & \text{on } \partial\Omega \cap \Omega_1 \\ U_1^{(k)} = U_2^{(k-1)} & \text{on } \Gamma \end{cases}$$

Subdomain 2

$$\begin{cases} -k_2 \Delta U_2^{(k)} = f & \text{in } \Omega_2 \\ U_2^{(k)} = 0 & \text{on } \partial\Omega \cap \Omega_2 \\ k_2 \dfrac{\partial u_2}{\partial n}^{(k)} + \gamma_2 U_2^{(k)} = k_1 \dfrac{\partial U_1}{\partial n}^{(\ell)} + \gamma_1 U_1^{(\ell)} & \text{on } \Gamma \end{cases}$$

$(\ell) : \ell = k-1$ (Jacobi)
$\ell = k$ (Gauss-Seidel).

b) Now we have the matrix version of the previous scheme (FE disc).

Subdomain 1 (D-D)

$$w=0 \text{ on } \Gamma_D$$

Weak form: $(\nabla w \cdot k \nabla u) - (\underset{=0}{\cancel{w}}, (k\nabla u)\cdot n)_\Gamma = (w, f)$

$w = N_i$

$u \simeq u^h = \sum_j N_j u_j$

$$k_{ij} = \int_{\Omega^{(e)}} \frac{dN_i}{dx}\frac{dN_j}{dx}$$

$$f_i = \int_{\Omega^{(e)}} \frac{dN_i}{dx} f .$$

$$\boxed{\begin{array}{ll} k\,\underline{\underline{k}}\,u_1^{(k)} = f & \text{in } \Omega_1 \\ u_1^{(k)} = 0 & \text{on } \partial\Omega \cap \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)} & \text{on } \Gamma \end{array}}$$

Subdomain 2 (R-D)

Weak form: $(\nabla w : k \nabla u_2^{(k)}) - (w, (k\nabla u)\cdot n)_\Gamma = (w, f)$

$$(\nabla w \cdot k \nabla u_2^{(k)}) - (w, (k_1 \nabla u_2^{(\ell)})\cdot n_1 + \gamma_1 \hat{u}_1^{(\ell)} - \gamma_2 u_2^{(k)}) = (w, f)$$

$$(\nabla w \cdot k \nabla u_2^{(k)}) + (w, \gamma_2 u_2^{(k)})_\Gamma = (w, f) + (w, \underbrace{(k_1 \nabla u_1^{(e)})\cdot n_1}_{\bar{q}_1})_\Gamma +$$

$$+ (w, \gamma_1 \bar{u}_1^{(k)})$$

$$(\nabla w \cdot k \nabla u^{(k)}) + (w, \gamma_2 u_2^{(k)})_\Gamma = (w, f) + (w, \bar{q}_1^{(\ell)})_\Gamma + (w, \gamma_1 \bar{u}_1^{(\ell)})$$

$w = N_i$

$u \simeq u^h = \sum_j N_j u_j$

$$k_{ij} = \int_{\Omega^{(e)}} \frac{dN_i}{dx}\frac{dN_j}{dx} \qquad q_i = \int_\Gamma N_i \bar{q}_1^{(\ell)}$$

$$f_i = \int_{\Omega^{(e)}} N_i f$$

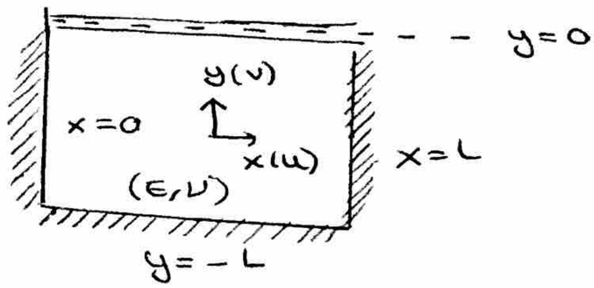
$$B_{ij} = \int_\Gamma N_i^{②} N_j^{②}$$

$$\boxed{\begin{array}{l} k\,\underline{\underline{k}}\,\underline{u}_2^{(k)} + \gamma_2 B u_{2,\Gamma}^{(k)} = \underline{f} + \underline{q}^{(2)} + \gamma_1 \hat{B}\,\bar{u}_1^{(\ell)} \\ u_2^{(k)} = 0 \quad \text{on } \partial\Omega \cap \Omega_2 \end{array}} \qquad \hat{B}_{ij} = \int_\Gamma N_i^{②} N_j^{①}$$

NOTE: Part c and d, I do not know how to solve.

3. Coupling of heterogeneous problems

E1)) Exercise 1



Euler-Bernoulli clamped beam:

$$EI\frac{d^4v}{dx^4} = f$$

$$v(0) = v(L) = 0$$

$$\frac{dv(0)}{dx} = \frac{dv(L)}{dx} = 0$$

a) Wall equations (plane stress behaviour).

$\underline{\underline{\sigma}} = \underline{\underline{D}}\,\underline{\underline{\varepsilon}}$ constitutive equation

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$d_{11} = d_{22} = \frac{E}{1-\nu^2}$$

$$d_{12} = d_{21} = \frac{\nu E}{1-\nu^2}$$

$$d_{33} = \frac{E}{2(1+\nu)}$$

the strains are related with the displacements via the kinematic eq's:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Equilibrium equation (First Cauchy motion equation):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \cancel{b_x}^{\,0} = \rho_0 \cancel{\frac{\partial^2 u_x}{\partial t^2}}^{\,0} \quad \text{quasi static problem} \longrightarrow$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho_0 \cancel{b_y}^{\,0} = \rho_0 \cancel{\frac{\partial^2 u_y}{\partial t^2}}^{\,0} \quad ''$$

body forces are not considered.

$$\longrightarrow \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Finally, when we combine the previous equations we obtain:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial x}\left[\left(\frac{E}{1-\nu^2}\right)\frac{\partial u}{\partial x}+\left(\frac{\nu E}{1-\nu^2}\right)\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[\left(\frac{E}{2(1+\nu)}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]=0\\&\frac{\partial}{\partial x}\left[\left(\frac{E}{2(1+\nu)}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\left(\frac{E}{1-\nu^2}\right)\frac{\partial v}{\partial y}+\left(\frac{\nu E}{1-\nu^2}\right)\frac{\partial u}{\partial x}\right]=0\end{aligned}}$$

with boundary conditions:

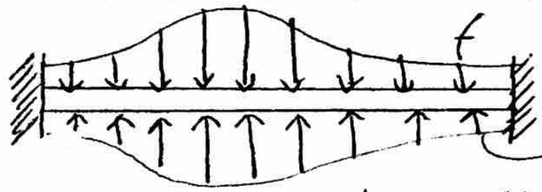
clamped lateral edge BC.

$$\begin{cases} u(0,y)=u(L,y)=0\\ v(0,y)=v(L,y)=0\\ \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \quad \text{on } (x=0,y) \text{ and } (x=L,y)\end{cases}$$

clamped bottom edge BC.

$$\begin{cases} u(x,-L)=0\\ v(x,-L)=0\\ \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \quad \text{on } (x, y=-L)\end{cases}$$

b) The beam over the elastic wall can be considered as a beam over an elastic foundation, thus:



Elastic reaction: $= k\,v$ (spring)

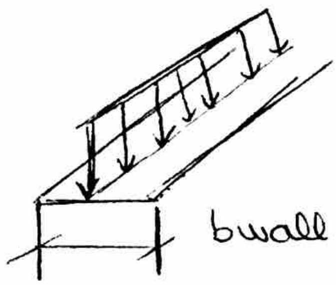
k is a modulus of the wall, for instance, the elastic modulus can be considered.

$$EI\frac{d^4v}{dx^4}=f-E_{wall}\,v \longrightarrow \boxed{\begin{cases}EI\frac{d^4v}{dx^4}+E_{wall}\,v=f\\ v(0)=v(L)=0\\ \frac{dv(0)}{dx}=\frac{dv(L)}{dx}=0\end{cases}}$$

c) The vertical displacement of the beam equals the vertical displacement imposed in the top edge of the wall:

$V_{wall}(x,y=0)=V_{beam}(x)$

For the normal component of the traction (σ_y), the elastic reation of the beam can be distributed in all the thickness of the wall.



$$\sigma_y(x, y=0) = \frac{-E_{wall} \cdot v_{beam}(x)}{b_{wall}}$$

uniform distribution of the pressure is supposed in the thickness.

d) In this case, a free-slip condition has been assumed in the contact between beam and wall. Hence there is no link in the horizontal displacements neither in the tangential component.

$$\boxed{\begin{array}{l} u_{wall}(x, y=0) = \text{free} \\ \tau_{xy}^{wall} = 0 \end{array}}$$

If the tangential component is different than 0, the horizontal displ. in the bottom part of the beam cannot be computed as $\theta \cdot h/2$. As a consequence, the Euler-Bernoulli beam equation needs to be modified. For the case of considering the influence of the friction in the rotations. Euler-Bernoulli beam theory only considers vertical loads.

E2) Exercise 2.

S_D and S_S Dirichlet-to-Neumann operators for the Darcy and Stokes problems.

λ: normal velocity on Γ.

$$S_S(\lambda) = S_D(\lambda)$$

a) Discrete version of the previous equation (using FE)

Stokes problem

the weak form of the Stokes problem is:

$$\int_{\Omega_S} \nabla \delta \underline{u_S} : \nu \underline{\nabla} \underline{u_S} - \int_{\Omega_S} (\nabla \cdot \delta \underline{u_S}) p_S + \int_{\partial \Omega_S} ((p_S \underline{\underline{I}} - \nu \underline{\nabla} \underline{u_S}) \cdot \underline{n}) \cdot \delta \underline{u_S} =$$

$$= \int_{\Omega} \delta \underline{u_S} \cdot \underline{f}$$

$$u_s \doteq u_{s,h} = \sum_j \underline{N}_j \, u_{s_j}$$

$$\delta \underline{u_s} = \underline{N_i}$$

$$p_s \doteq p_{s,h} = \sum_j \varphi_j \, p_{s,j}$$

Applying the previous discretizations to the weak form:

$$K_{ij} = \int_{\Omega^{(e)}} \underline{\nabla} \, \underline{N_i} : \nu \underline{\nabla} N_j \; ; \qquad G_{ij} = \int_{\Omega^{(e)}} (\underline{\nabla} \cdot \underline{N_i}) \varphi_j \; ; \qquad B^{us}_{ij} = \int_{\partial\Omega^{(e)}} \underline{N_i} \, (\varphi_j \cdot \underline{n})$$

$$B^{ps}_{ij} = \int_{\partial\Omega^{(e)}} \underline{N_i} \cdot (n \cdot (-\nu \underline{\nabla} N_j))$$

$$\underline{\underline{K}} \, \underline{u_s} - \underline{\underline{G}} \, \underline{p_s} - \underline{\underline{B}}^{us} \underline{u_s} + \underline{\underline{B}}^{ps} \underline{p_s} = \underline{f}$$

<u>Darcy problem</u>

The weak form of the Darcy problem is:

$$\int_{\Omega_D} \delta \underline{u_D} \cdot (k^{-1} \underline{u_D}) + \int_{\partial\Omega_D} (\underline{\delta u_D} \, \varphi) \cdot \underline{\pi} - \int_{\Omega_D} (\nabla \cdot \underline{\delta u_D}) \varphi = 0$$

- $u_D \doteq u_{D,h} = \sum_j \underline{N}_j \, u_{D_j}$
- $\underline{\delta u_D} = N_i$
- $p_D \doteq p_{D,h} = \sum_j \varphi_j \, p_{D_j}$

The obtained matrices are:

$$M_{ij} = \int_{\Omega^{(e)}} \underline{N_i} \cdot (k^{-1} \underline{N_j}) \; ; \qquad B_{ij} = \int_{\Omega^{(e)}} (\underline{N_j} \, \varphi_j) \cdot \pi \; ; \qquad H_{ij} = \int_{\Omega^{(e)}} (\nabla \cdot \underline{N_i}) \varphi_j$$

$$\underline{\underline{M}} \, \underline{u_D} + \underline{\underline{B}}^{D} \varphi - \underline{\underline{H}} \, \varphi = 0$$

A^s_{11} $\quad A^s_{1r}$ $\quad A^D_{r1}$

$$A^s_{r1} \quad \begin{bmatrix} K_{11} & G_{11} & B_{1r} & 0 & 0 \\ G^T_{11} & 0 & G^T_{1r} & 0 & 0 \\ K_{r1} & G_{r1} & B_{rr} & M_{rr} & 0 \\ 0 & 0 & -M^T_{rr} & M_{11} & H \\ 0 & 0 & 0 & H & H_{11} \end{bmatrix} \begin{bmatrix} u_s \\ p_s \\ \lambda \\ u_D \\ \varphi \end{bmatrix} = \begin{bmatrix} \\ \\ \underline{f} \\ \\ \\ \end{bmatrix}$$

A^D_{1r} $\quad A^D_{11}$

Simplifying the representation of the system:

$$\begin{bmatrix} A_{II}^S & A_{I\Gamma}^S & 0 \\ A_{\Gamma I}^S & B_{\Gamma\Gamma} & A_{\Gamma I}^D \\ 0 & A_{I\Gamma}^D & A_{II}^D \end{bmatrix} \begin{bmatrix} x_S \\ \lambda \\ x_D \end{bmatrix} = \begin{bmatrix} f_S \\ f_\Gamma \\ f_D \end{bmatrix}$$

$$x_S = (A_{II}^S)^{-1}(f_S - A_{I\Gamma}^S \lambda)$$

$$x_D = (A_{II}^D)^{-1}(f_D - A_{I\Gamma}^D \lambda)$$

$$A_{\Gamma I}^S\left((A_{II}^S)^{-1}(f_S - A_{I\Gamma}^S\lambda)\right) + B_{\Gamma\Gamma}\lambda + A_{\Gamma I}^D\left((A_{II}^D)^{-1}(f_D - A_{I\Gamma}^D\lambda)\right) = f_\Gamma$$

$$-A_{\Gamma I}^S (A_{II}^S)^{-1} A_{I\Gamma}^S \lambda + B_{\Gamma\Gamma}\lambda - A_{\Gamma I}^D (A_{II}^D)^{-1} A_{I\Gamma}^D \lambda = f_\Gamma - A_{\Gamma I}^S (A_{II}^S)^{-1} f_S -$$

$$- A_{\Gamma I}^D (A_{II}^D)^{-1} f_D$$

$$\boxed{\left(B_{\Gamma\Gamma} - A_{\Gamma I}^S (A_{II}^S)^{-1} A_{I\Gamma}^S - A_{\Gamma I}^D (A_{II}^D)^{-1} A_{I\Gamma}^D\right)\lambda = f_\Gamma - A_{\Gamma I}^S (A_{II}^S)^{-1} f_S - A_{\Gamma I}^D (A_{II}^D)^{-1} f_D}$$

b) Dirichlet - Neumann IBS

First subdomain (Stokes)

$$\begin{bmatrix} K_{II} & G_{II} & B_{I\Gamma} \\ G_{II}^T & 0 & G_{I\Gamma} \\ K_{\Gamma I} & G_{\Gamma I} & B_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_S^{(k)} \\ P_S^{(k)} \\ \lambda^{(k)} \end{bmatrix} = \begin{bmatrix} f_S \\ 0 \\ f_\Gamma - \Pi_{\Gamma\Gamma} U_D^{(k-1)} \end{bmatrix}$$

Second subdomain (Darcy)

$$\begin{bmatrix} \Pi_{II} & H \\ H^T & H_{II} \end{bmatrix} \begin{bmatrix} U_D^{(k)} \\ \varphi^{(k)} \end{bmatrix} = \begin{bmatrix} -\Pi_{\Gamma\Gamma}\lambda^{(k-1)} \\ 0 \end{bmatrix}$$

4. Monolithic and partitioned schemes in time (t)

1D transient heat transfer equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1] \\ u(x=0,t)=0 \\ u(x=1,t)=0 \\ u(x,t=0)=0 \end{cases}$$

E1) Exercise 1.

Space disc → FE (linear elements)

Time discretization → BDF1.

$(k=1,\ f=1,\ \delta t=1)$

- weak form of the problem:

$$\int_\Omega v u_t - k\int_\Omega v u_{xx} = \int_\Omega v f \qquad \forall v \in H_0^1$$

$$\int_\Omega v u_t + k\int_\Omega u_x v_x - k\,\cancel{(v u_x)}\Big|_0^1 = \int_\Omega v f$$

since $v=0$ on Γ_D.

$$(v, u_t)_\Omega + k(v_x, u_x)_\Omega = (v, f)_\Omega$$

- Space discretization:

$$v \equiv N_i(x)$$

$$u \simeq u_h = \sum_j N_j(x) u_j$$

$$\underbrace{(N_i, N_j u_{j,t})_\Omega}_{\underline{\underline{M}}} + \underbrace{k\left(\frac{dN_i}{dx}, \frac{dN_j}{dx}u_j\right)_\Omega}_{\underline{\underline{K}}} = \underbrace{(N_i(x), f)_\Omega}_{\underline{f}}$$

$$M_{ij} = \int_{\Omega^{(e)}} N_i N_j \qquad K_{ij} = \int_{\Omega^{(e)}} \cancel{k}^{\,1} \frac{dN_i}{dx}\frac{dN_j}{dx} \qquad f_i = \int_{\Omega^{(e)}} N_i \cancel{f}^{\,1}$$

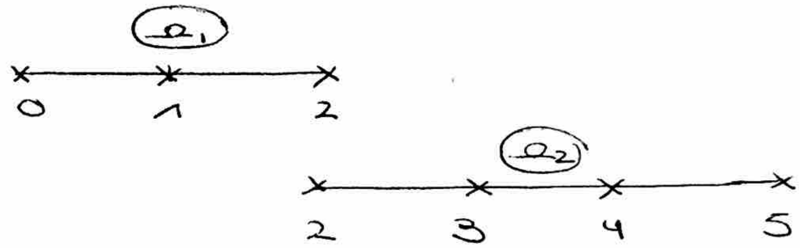
Before time discretization: $\underline{\underline{\Pi}}\,\underline{u}_t + \underline{\underline{K}}\,\underline{u} = \underline{f}$

- time discretization (BDF1): $u_t = \dfrac{u^{n+1} - u^n}{\delta t}$

$$\underline{\underline{\Pi}}\,\frac{(\underline{u}^{n+1} - \underline{u}^n)}{\cancel{\delta t}} + \underline{\underline{K}}\,\underline{u}^{n+1} = \underline{f}$$

$$\boxed{(\underline{\underline{\Pi}} + \underline{\underline{K}})\,\underline{u}^{n+1} = \underline{f} + \underline{\underline{\Pi}}\,\underline{u}^n} \qquad K = \qquad \delta t = 1$$

E2) Exercise 2:



$h = 0'2$

Show that if a monolithic approach is adopted, no boundary integrals are required at the interface.

Subdomain 1:

$$\begin{cases} \displaystyle\int_0^{0'4} v u_t^{(1)} - k\,(v u_x^{(1)})\Big|_{\cancel{0}}^{0'4} + k\int_0^{0'4} u_x^{(1)} v_x = \int_0^{0'4} v f \quad \text{in } \Omega_1 \\ (v = 0 \text{ on } \Gamma_D) \\ u = 0 \quad \text{on } \Gamma_D\ (x=0) \\ u(x, t=0) = 0 \quad \text{on } \Omega_1 \\ u^{(1)} = u^{(2)} \quad \text{on } \Gamma\ (x = 0'4) \\ -k\,(v u_x^{(1)}) = +k\,(v u_x^{(2)}) \text{ on } \Gamma \end{cases}$$

Subdomain 2:

$$\begin{cases} \displaystyle\int_{0'4}^{1} v u_t^{(2)} - k\,(v u_x^{(2)})\Big|_{0'4}^{\cancel{1}} + k\int_{0'4}^{1} u_x^{(2)} v_x = \int_{0'4}^{1} v f \quad \text{in } \Omega_2 \\ 0\ (v = 0 \text{ on } \Gamma_D) \\ u = 0 \quad \text{on } \Gamma_D\ (x=1) \\ u(x, t=0) = 0 \quad \text{on } \Omega_2 \\ u^{(2)} = u^{(1)} \quad \text{on } \Gamma \\ -k\,(v u_x^{(2)}) = +k\,(v u_x^{(1)}) \text{ on } \Gamma \end{cases}$$

$(v, u_t^{(1)})_{\Omega_1} - k\,(v, u_x^{(1)})_\Gamma + k\,(u_x^{(1)}, v_x)_{\Omega_1} = (v, f)_{\Omega_1} \rightarrow$ subdomain 1

$(v, u_t^{(2)})_{\Omega_2} - k\,(v, u_x^{(2)})_\Gamma + k\,(u_x^{(2)}, v_x)_{\Omega_2} = (v, f)_{\Omega_2} \rightarrow$ subdomain 2.

If both equations are added:

$$(v, u_t^{(1)})_{\Omega_1} + (v, u_t^{(2)})_{\Omega_2} \underbrace{- k\,(v, u_x^{(1)})_\Gamma - k\,(v, u_x^{(2)})_\Gamma}_{} + k\,(u_x^{(1)}, v_x)_{\Omega_1} +$$

$$+ k\,(u_x^{(2)}, v_x)_{\Omega_2} = (v, f)_{\Omega_1} + (v, f)_{\Omega_2}$$

If the subdomains have the same interpolation, it can be directly applied that:

$-k(v u_x^{(1)}) = +k(v u_x^{(2)})$

$$(v, u_t^{(1)})_{\Omega_1} + (v, u_t^{(2)})_{\Omega_2} - k\,(v, u_x^{(2)})_\Gamma + k(v, u_x^{(2)})_\Gamma + k\,(u_x^{(1)}, v_x)_{\Omega_1} +$$

$$+ k\,(u_x^{(2)}, v_x)_{\Omega_2} = (v, f)_{\Omega_1} + (v, f)_{\Omega_2}$$

Considering that $\Omega = \Omega_1 \cup \Omega_2$ the original form of the problem is recovered:

$$\boxed{(v, u_t)_\Omega + k\,(u_x, v_x)_\Omega = (v, f)_\Omega} \rightsquigarrow$$ No boundary integrals appear at the interface.

E3) Algebraic form of the Dirichlet-to-Neumann op (u_i^n, u_2^{n+1} known)

For the first subdomain (DD problem) the system can be written in terms of the interior and boundary components as follows.

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)\,n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} f_I^1 \\ f_\Gamma^1 \end{bmatrix} + \begin{bmatrix} \Pi_{II}^{(1)} & \Pi_{I\Gamma}^{(1)} \\ \Pi_{\Gamma I}^{(1)} & \Pi_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)\,n} \\ u_\Gamma^{(1)\,n} \end{bmatrix}$$

Dirichlet cond at the interface node.

$\underline{\underline{A}} = \underline{\underline{K}} + \underline{\underline{\Pi}}$

$$A_{II}^{(1)} u_1^{(1)\,n+1} + A_{I\Gamma}^{(1)} u_2^{n+1} = f_1^{(1)} + \Pi_{II}^{(1)} u_1^{(1)\,n} + \Pi_{I\Gamma}^{(1)} u_\Gamma^{(1)\,n}.$$

$$\boxed{u_1^{(1)\,n+1} = (A_{II}^{(1)})^{-1} \left(f_1^{(1)} + \Pi_{II}^{(1)} u_1^{(1)\,n} + \Pi_{I\Gamma}^{(1)} u_\Gamma^{(1)\,n} - A_{I\Gamma}^{(1)} u_2^{n+1}\right)}$$

E4) Algebraic form of the Neumann-to Dirichlet operator

(U_I^{n}, ϕ_2^{n+1} known)

$$\begin{bmatrix} A_{\Gamma\Gamma}^{(2)} & A_{\Gamma I}^{(2)} \\ A_{I\Gamma}^{(2)} & A_{II}^{(2)} \end{bmatrix} \begin{bmatrix} U_\Gamma^{(2)\,n+1} \\ U_I^{(2)\,n+1} \end{bmatrix} = \begin{bmatrix} \phi_2^{n+1} \\ 0 \end{bmatrix} + \begin{bmatrix} f_\Gamma^{(2)} \\ f_I^{(2)} \end{bmatrix} + \begin{bmatrix} \Pi_{\Gamma\Gamma}^{(2)} & \Pi_{\Gamma I}^{(2)} \\ \Pi_{I\Gamma}^{(2)} & \Pi_{II}^{(2)} \end{bmatrix} \begin{bmatrix} U_\Gamma^{(2)\,n} \\ U_I^{(2)\,n} \end{bmatrix}$$

From the first eq:

$$(*)\quad U_\Gamma^{(2)\,n+1} = (A_{I\Gamma}^{(2)})^{-1}\left(\phi_2^{n+1} + f_\Gamma^{(2)} + \Pi_{\Gamma\Gamma}^{(2)} U_\Gamma^{(2)\,n} + \Pi_{\Gamma I}^{(2)} U_I^{(2)\,n} - A_{\Gamma I}^{(2)} U_I^{(2)\,n+1}\right)$$

For the second:

$$A_{I\Gamma}^{(2)}\left((A_{\Gamma\Gamma}^{(2)})^{-1}\left(\phi_2^{n+1} + f_\Gamma^{(2)} + \Pi_{\Gamma\Gamma}^{(2)} U_\Gamma^{(2)\,n} + \Pi_{\Gamma I}^{(2)} U_I^{(2)\,n} - A_{\Gamma I}^{(2)} U_I^{(2)\,n+1}\right)\right) +$$

$$+ A_{II}^{(2)} U_I^{(2)\,n+1} = f_I^{(2)} + \Pi_{II}^{(2)} U_I^{(2)\,n} + \Pi_{I\Gamma}^{(2)} U_\Gamma^{(2)\,n}$$

$$\left(A_{II}^{(2)} - A_{I\Gamma}^{(2)} (A_{\Gamma\Gamma}^{(2)})^{-1} A_{\Gamma I}^{(2)}\right) U^{(2)\,n+1} = f_I^{(2)} + \Pi_{II}^{(2)} U_I^{(2)\,n} + \Pi_{I\Gamma}^{(2)} U_\Gamma^{(2)\,n} -$$

$$- A_{I\Gamma}^{(2)} (A_{\Gamma\Gamma}^{(2)})^{-1}\left(f_\Gamma^{(2)} + \Pi_{\Gamma\Gamma}^{(2)} U_\Gamma^{(2)\,n} + \Pi_{\Gamma I}^{(2)} U_I^{(2)\,n} + \phi_2^{n+1}\right)$$

$$\boxed{\begin{aligned} U_I^{(2)\,n+1} = {} & \left(A_{II}^{(2)} - A_{I\Gamma}^{(2)}(A_{\Gamma\Gamma}^{(2)})^{-1} A_{\Gamma I}^{(2)}\right)^{-1}\Big(f_I^{(2)} + \Pi_{II}^{(2)} U_I^{(2)\,n} + \Pi_{I\Gamma}^{(2)} U_\Gamma^{(2)\,n} - \\ & - A_{I\Gamma}^{(2)} (A_{\Gamma\Gamma}^{(2)})^{-1}\left(\phi^{n+1} + \Pi_{\Gamma\Gamma}^{(2)} U_\Gamma^{(2)\,n} + \Pi_{\Gamma I}^{(2)} U_\Gamma^{(2)\,n} + f_\Gamma^{(2)}\right)\Big) \end{aligned}}$$

When we have computed the interior nodes, the solution at the interface can be recovered with (*)

In the exercise $\underline{\underline{A}} = \underline{\underline{\Pi}} + \underline{\underline{K}}$ as was stated in the previous one.

E5) Iterative algorithm for a staggered approach (Left sbd: D-D; Right sbd: N-D)

Representative scheme of the problem:

$$\begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & 0 \\ A_{10} & A_{11} & A_{12} & 0 & 0 & 0 \\ 0 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & A_{33} & A_{34} & 0 \\ 0 & 0 & 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} u \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

A_{21} is the coupling term 2nd sbd system

1rst sbd system

A_{23} is the coupling term

If we particularize to this problem:

Subdomain 1

$$\left(\begin{bmatrix} A_{00} & A_{01} & 0 \\ A_{10} & A_{11} & A_{12} \\ 0 & A_{21} & A_{22}\end{bmatrix} + \begin{bmatrix} \Pi_{00} & \Pi_{01} & 0 \\ \Pi_{10} & \Pi_{11} & \Pi_{12} \\ 0 & \Pi_{21} & \Pi_{22}\end{bmatrix}\right)\begin{bmatrix} 0 \\ U_1^{n+1,i} \\ U_2^{(2)\,n+1,i-1}\end{bmatrix} =$$

Dirichtel BC with the interf. values of the previous it in sub (2).

$$= \begin{bmatrix} f_0 \\ f_1 \\ f_2 - A_{23}U_3^{n+1(i-1)} - \Pi_{23}U_3^{n+1(i-1)} \end{bmatrix} + \begin{bmatrix} \Pi_{00} & \Pi_{01} & 0 \\ \Pi_{10} & \Pi_{11} & \Pi_{12} \\ 0 & \Pi_{21} & \Pi_{22}\end{bmatrix}\begin{bmatrix} 0 \\ U_1^n \\ U_2^n\end{bmatrix}$$

Approximation of the coupling terms with the values of the previous iteration in sub. (2).

When we solve this system, the flux to be imposed in the next iteration of subdomain 2 can be obtained ($\phi_2^{(2)\,n+1,i+1} = \phi_2^{(1)\,n+1,i}$)

Subdomain 2

$$\left(\begin{bmatrix} A_{22} & A_{23} & 0 & 0 \\ A_{32} & A_{33} & A_{34} & 0 \\ 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & A_{54} & A_{55}\end{bmatrix} + \begin{bmatrix} \Pi_{22} & \Pi_{23} & 0 & 0 \\ \Pi_{32} & \Pi_{33} & \Pi_{34} & 0 \\ 0 & \Pi_{43} & \Pi_{44} & \Pi_{45} \\ 0 & 0 & \Pi_{54} & \Pi_{55}\end{bmatrix}\right)\begin{bmatrix} U_2^{n+1,i} \\ U_3^{n+1,i} \\ U_4^{n+1,i} \\ 0\end{bmatrix} =$$

→ approxim. of the coupling terms with the values of the previous it. in sub (1)

$$= \begin{bmatrix} f_2 - A_{21}\hat{U}_1^{n+1,(i-1)} - \Pi_{21}\hat{U}_1^{n+1,(i-1)} \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} + \begin{bmatrix} \Pi_{22} & \Pi_{23} & 0 & 0 \\ \Pi_{32} & \Pi_{33} & \Pi_{34} & 0 \\ 0 & \Pi_{43} & \Pi_{44} & \Pi_{45} \\ 0 & 0 & \Pi_{54} & \Pi_{55}\end{bmatrix}\begin{bmatrix} U_2^n \\ U_3^n \\ U_4 \\ 0\end{bmatrix}$$

$$+ \begin{bmatrix} -\phi_2^{(1)\,n+1,(i-1)} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Neumann BC, with the flux obtained at the previous iteration in sub. (1).

E6) Iterative algorithm for a substitution and iteration by sled scheme.

Subdomain 1 For the first sub. the scheme remains as in exercise 5.

Subdomain 2

$$\left[\begin{bmatrix} A_{22} & A_{23} & 0 & 0 \\ A_{32} & A_{33} & A_{43} & 0 \\ 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & A_{54} & A_{55} \end{bmatrix} + \begin{bmatrix} \Pi_{22} & \Pi_{23} & 0 & 0 \\ \Pi_{32} & \Pi_{33} & \Pi_{34} & 0 \\ 0 & \Pi_{43} & \Pi_{44} & \Pi_{45} \\ 0 & 0 & \Pi_{54} & \Pi_{55} \end{bmatrix}\right] \begin{bmatrix} U_2^{n+1,i} \\ U_3^{n+1,i} \\ U_4^{n+1,i} \\ 0 \end{bmatrix} =$$

the coupling term are substituded instead of approximated.

$$= \begin{bmatrix} f_2 - A_{21} U_1^{n+1(i)} - \Pi_{21} U_1^{n+1(i)} \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} + \begin{bmatrix} \Pi_{22} & \Pi_{23} & 0 & 0 \\ \Pi_{32} & \Pi_{33} & \Pi_{34} & 0 \\ 0 & \Pi_{43} & \Pi_{44} & \Pi_{45} \\ 0 & 0 & \Pi_{54} & \Pi_{55} \end{bmatrix} \begin{bmatrix} U_2^{n} \\ U_3^{n} \\ U_4^{n} \\ 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} -\phi_2^{(2)\, n+1,i} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ → Substituting, we obtain the Neuman BC.

<u>E7</u>) Algebraic system of the left subdomain (D-D) applying the boundary conditions with Nitche's method. How is the variation of "k" according to "α"?

$$\begin{cases} \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } [0, 0'4] \\ & u(x=0,t) = 0 \\ & u(x=0'4,t) = \bar{u}_2^{(2)} \\ & u(x,t=0) = 0 \end{cases}$$

The weak form of the problem:

$$(w, U_t)_\Omega + \cancel{k}\hat{}\,(w_x, U_x)_\Omega - \cancel{k}\hat{}\,(w, U_x)_\Gamma = (w, f)_\Omega$$

Applying Nitcho's method:

$$(w, U_t)_\Omega + (w_x, U_x)_\Omega - \hat{\cancel{k}}(w, U_x)_\Gamma + \alpha (w, U)_\Gamma - (w_x, U)_\Gamma =$$

$$= (w, f)_\Omega + \alpha (w, \bar{u})_\Gamma - \hat{\cancel{k}}(w_x, \bar{u})_\Gamma$$

$(w, u_t)_\Omega + (w_x, u_x)_\Omega - (w, u_x)_\Gamma + \alpha (w, u)_\Gamma - (w_x, u)_\Gamma =$

$= (w, f)_\Omega + \alpha \cancel{(w, 0)}_\Gamma - \cancel{(w_x, 0)}_\Gamma + \alpha (w, \bar{u}_2^{②})_\Gamma -$

$- (w_x, \bar{u}_2^{②})_\Gamma$

we apply BDF1 for discretization in time:

$$u_t = \frac{u^{n+1} - u^n}{\Delta t}$$

$(w, u^{n+1})_\Omega + (w, u_x^{n+1})_\Omega - (w, u_x^{n+1})_\Omega + \alpha (w, u^{n+1})_\Gamma - (w_x, u^{n+1})_\Gamma =$

$= (w, f)_\Omega + \alpha (w, \bar{u}_2^{②})_{0'4} - (w_x, \bar{u}_2^{②})_{0'4}$

$w = N_i$ Using Galerkin method

$u \simeq u^h = \sum_j N_j u_j$

$(w, u_t) \Longrightarrow M_{ij} = \int_{\Omega^{(e)}} N_i N_j$; $(w, u)_\Gamma \Rightarrow B^2 = N_i N_j \Big|_0^{0'4}$

$(w_x, u_x) \Rightarrow A_{ij} = \int_{\Omega^{(e)}} \frac{dN_i}{dx} \frac{dN_j}{dx}$; $(w_x, u)_\Gamma \Longrightarrow B^3 = \frac{dN_i}{dx} N_j \Big|_0^{0'4}$

$(w, u_x)_\Gamma \Longrightarrow B^1 = N_i \frac{dN_j}{dx} \Big|_0^{0'4}$; $(w, f) \Longrightarrow f_i = \int_{\Omega^{(e)}} N_i f$

So, the system to be solved is:

$$\boxed{\begin{aligned} &(\underline{\underline{M}} + \underline{\underline{A}})\, u^{n+1} - B^1 u^{n+1} \Big|_0^{0'4} + \alpha B^2 u \Big|_0^{0'4} - B_3 u^{n+1} \Big|_0^{0'4} = \\ &= f + \underline{\underline{M}}\, u^n + \alpha B_2 \bar{u}_2^{② n+1} \Big|_{0'4} - B_3 \bar{u}_2^{② n+1} \Big|_{0'4} \end{aligned}}$$

we have to select "α" properly for avoiding an ill-conditioned matrix. Despite the fact that the matrix obtained with Nitsche's method is better conditioned than the one obtained with penalty method, it has a greater condition number than the one with strong imposition of Dirichlet boundary conditions.

5. Operator splitting techniques

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f \quad \text{in } [0,1]$$

$$\begin{cases} u(x=0,t)=0 \\ u(x=1,t)=0 \\ u(x,t=0)=0 \end{cases}$$

with $k=1$; $a_x = 1$; $f=1$

E1) Space discret : FE *

Time discret : BDF1

* we have solved the problem with 3 elements instead of 2 since the problem solved with 2 elements does not present convection the central term of the $\underline{\underline{a}}$ matrix is 0.

$$u_t - u_{xx} + u_x = 1$$

$$(w, u_t)_\Omega - \underbrace{(w, u_{xx})_\Omega} + (w, u_x)_\Omega = (w, 1)_\Omega$$ weighted residual form

$$(w, u_{xx})_\Omega = w u_x \Big|_0^1 - (w_x, u_x)_\Omega$$ if we integrate by parts

$$(w, u_t)_\Omega + (w_x, u_x)_\Omega - \cancel{w u_x \Big|_0^1}^{\,0} + (w, u_x)_\Omega = (w, 1)_\Omega$$

selecting w such that $w(0) = w(1) = 0$.

<u>Space discretitation</u> :

$$w = N_i$$
$$u \doteq u_h = \sum_j N_j u_j$$

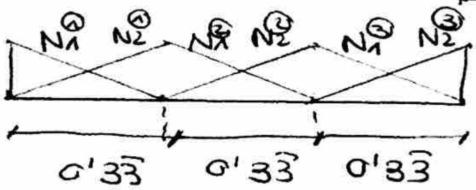
$$\sum_j \Big[\underbrace{\Big(N_i, N_j \frac{du_j}{dt}\Big)_\Omega}_{\underline{\underline{M}}} + \underbrace{\Big(\frac{dN_i}{dx}, \frac{dN_j}{dx} u_j\Big)_\Omega}_{\underline{\underline{K}}} + \underbrace{\Big(N_i, \frac{dN_j}{dx} u_j\Big)_\Omega}_{\underline{\underline{a}}} \Big] =$$

$$= \sum_j \underbrace{(N_i, 1)_\Omega}_{\underline{f}} \, ; \qquad \underline{\underline{M}}\,\underline{u}_t + (\underline{\underline{K}} + \underline{\underline{a}})\underline{u} = \underline{f}$$

Time discretization (BDF1)

$$u_t = \frac{u^{n+1}-u^{n}}{\Delta t}$$

$$\boxed{\frac{\underline{\underline{M}}}{\Delta t}\,\underline{u}^{n+1} + (\underline{\underline{K}}+\underline{\underline{a}})\,\underline{u}^{n+1} = \underline{f} + \frac{\underline{\underline{M}}}{\Delta t}\,\underline{u}^{n}}$$



$$N_1 = \frac{x_2 - x}{\ell^{(e)}} \quad \left(\frac{dN_1}{dx} = \frac{-1}{\ell}\right)$$

$$N_2 = \frac{x - x_1}{\ell^{(e)}} \quad \left(\frac{dN_2}{dx} = \frac{1}{\ell}\right)$$

$$\underline{\underline{M}}^{(1)} = \underline{\underline{M}}^{(2)} = \underline{\underline{M}}^{(3)} = \int_0^{0'3\hat{3}} \begin{bmatrix} \frac{0'\hat{3}-x}{0'\hat{3}} \cdot \frac{0'\hat{3}-x}{0'\hat{3}} & \frac{0'\hat{3}-x}{0'\hat{3}} \frac{x}{0'\hat{3}} \\ \frac{x}{0'\hat{3}} \frac{0'\hat{3}-x}{0'\hat{3}} & \frac{x}{0'\hat{3}} \cdot \frac{x}{0'\hat{3}} \end{bmatrix} = \begin{bmatrix} 1/9 & 1/18 \\ 1/18 & 1/9 \end{bmatrix}$$

$$\underline{\underline{K}}^{(1)} = \underline{\underline{K}}^{(2)} = \underline{\underline{K}}^{(3)} = \int_0^{0'3\hat{3}} \begin{bmatrix} -1/0'3\hat{3} \cdot -1/0'3\hat{3} & -1/0'3\hat{3} \cdot \frac{1}{0'3\hat{3}} \\ 1/0'3\hat{3} \cdot -1/0'3\hat{3} & 1/0'3\hat{3} \cdot 1/0'3\hat{3} \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\underline{\underline{a}}^{(1)} = \underline{\underline{a}}^{(2)} = \underline{\underline{a}}^{3} = \int_0^{0'3\hat{3}} \begin{bmatrix} \frac{0'\hat{3}-x}{0'\hat{3}} \cdot \frac{-1}{0'\hat{3}} & \frac{0'\hat{3}-x}{0'\hat{3}} \frac{1}{0'\hat{3}} \\ \frac{x}{0'3} \cdot \frac{-1}{0'\hat{3}} & \frac{x}{0'\hat{3}} \cdot \frac{1}{0'\hat{3}} \end{bmatrix} = \begin{bmatrix} -0'5 & 0'5 \\ -0'5 & 0'5 \end{bmatrix}$$

$$f^{(1)} = f^{(2)} = f^{(3)} = \int_0^{0'\hat{3}} \begin{matrix} \frac{0'\hat{3}}{0'\hat{3}} \\ \frac{x}{0'\hat{3}} \end{matrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

So the assembled system of eq:

$$\left[\frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} + \right.$$

$$\left. + \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}\right] \begin{bmatrix} 0 \\ u_2^{n+1} \\ u_3^{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix} + \frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & \frac{1}{18} \\ 0 & 0 & 1/18 & \frac{1}{9} \end{bmatrix}\begin{bmatrix} 0 \\ u_2^{n} \\ u_3^{n} \\ 0 \end{bmatrix}$$

When we solve the first time step (initial condition $\underline{u}=0$) we obtain:

$$u_2 = \frac{6\Delta t\,(51\,\Delta t+1)}{2943\,\Delta t^2+324\,\Delta t+5}$$

$$u_3 = \frac{6\Delta t\,(15\Delta t+1)}{2943\,\Delta t^2+324\,\Delta t+5}$$

E2) First order operator splitting technique:

$$\left.\begin{aligned} &u_t - u_{xx} + u_x = 1 \\ &\alpha_\nu u = \frac{d^2u}{dx^2} \\ &\alpha_a u = \frac{du}{dx} \end{aligned}\right\}$$

First solve:

$$\frac{du_a}{dt} + \frac{du_a}{dx} = 0$$
$$u_a(t_n) = u^n$$

Second:

$$\frac{du}{dt} + \frac{d^2u}{dx^2} = f \;;\; u(t_{n+1}) = u_a(t_{n+1})$$

So at the algebraic level:

First: $\left(\frac{\underline{\underline{M}}}{\Delta t} + \underline{\underline{a}}\right)\underline{u_a}^{n+1} = \frac{\underline{\underline{M}}}{\Delta t}\underline{u}^n$; second $\left(\frac{\underline{\underline{M}}}{\Delta t}+K\right)\underline{u}^{n+1} = \frac{\underline{\underline{M}}}{\Delta t}u_a^{n+1} + f$

<u>First step</u>:

$$\left[\frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix} + \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}\right]\begin{bmatrix} 0 \\ u_{a,2}^{n+1} \\ u_{a,3}^{n+1} \\ 0 \end{bmatrix} = \frac{1}{\Delta t}\,\cdot$$

$$\cdot\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix}\begin{bmatrix} 0 \\ u_2^n \\ u_3^n \\ 0 \end{bmatrix} \begin{matrix} \\ = 0 \text{ initial cond} \\ = 0 \text{ initial cond} \\ \\ \end{matrix} \to 0$$

$$\boxed{\begin{aligned} u_{a,2}^{n+1} &= 0 \\ u_{a,3}^{n+1} &= 0 \end{aligned}}$$

Second step.

$$\left[\frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/18 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}}_{=0}\right]\begin{bmatrix} 0 \\ u_2^{n+1} \\ u_3^{n+1} \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\Delta t}\begin{bmatrix} 1/9 & 1/18 & 0 & 0 \\ 1/18 & 2/9 & 1/18 & 0 \\ 0 & 1/18 & 2/9 & 1/18 \\ 0 & 0 & 1/18 & 1/9 \end{bmatrix}\begin{bmatrix} 0 \\ \cancel{u_{a12}^{n+1}} = 0 \\ \cancel{u_{a13}^{n+1}} = 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}$$

$$u_2^{n+1} = 6\Delta t / (54\Delta t + 5)$$

$$u_3^{n+1} = 6\Delta t / (54\Delta t + 5)$$

E3) Error assessment.

We have to know that for the operator splitting tech. solution the solution is the same in both nodes. This can be view as an error associated to the fact that initial solution is 0 and that the source term has been applied in the second step of the splitting operation. No transport occurs in the 1st step and the solution is symmetric:

		Node ②	③
Error:	$\Delta t = 1$	[-0'0063	, 0'0047]
	$\Delta t = 0'5$	[-0'0057	, 0'0043]
	$\Delta t = 0'25$	[-0'0047	, 0'0037]

$$\text{error} = u^{n+1} - u^{n+1}_{splitted}$$

Because we do not have transport in the splitted solution, we have an excessive value in node 2, meanwhile it is under the expected in node 3. It can be seen that the error is of the form $O(\Delta t)$ and infralinear

6. Fractional step methods

Yosida Scheme (Incompressible Navier-Stokes eq's)

1. $M \frac{1}{\Delta t}(\hat{U}^{n+1} - U^n) + K\hat{U}^{n+1} = f - G\tilde{P}^{n+1}$

2. $DM^{-1}GP^{n+1} = \frac{1}{\Delta t} D\hat{U}^{n+1} - DM^{-1}G\tilde{P}^{n+1}$

3. $M\frac{1}{\Delta t}(U^{n+1} - \hat{U}^{n+1}) + \alpha K(U^{n+1} - \hat{U}^{n+1}) + G(P^{n+1} - \hat{P}^{n+1}) =$

$= 0$

E1) optimal α parameter?

3. $\frac{M}{\Delta t}U^{n+1} - \frac{M}{\Delta t}\hat{U}^{n+1} + \alpha K U^{n+1} - \alpha K\hat{U}^{n+1} + GP^{n+1} - G\hat{P}^{n+1} = 0$

1. $\frac{M}{\Delta t}\hat{U}^{n+1} - \frac{M}{\Delta t}U^n + K\hat{U}^{n+1} + G\hat{P}^{n+1} = f$

so with these 2 equations I have above:

$$\frac{M}{\Delta t}U^{n+1} - \cancel{\frac{M}{\Delta t}\hat{U}^{n+1}} + \cancel{\frac{M}{\Delta t}\hat{U}^{n+1}} - \frac{M}{\Delta t}U^n + \alpha K U^{n+1} - \alpha K\hat{U}^{n+1} +$$

$$+ K\hat{U}^{n+1} + GP^{n+1} - \cancel{G\hat{P}^{n+1}} + \cancel{G\hat{P}^{n+1}} = f$$

$$\frac{M}{\Delta t}U^{n+1} - \frac{M}{\Delta t}U^n + \alpha K U^{n+1} + (1-\alpha)K\hat{U}^{n+1} + GP^{n+1} = f$$

$$\frac{M}{\Delta t}(U^{n+1} - U^n) + \alpha K U^{n+1} + (1-\alpha)K\hat{U}^{n+1} + GP^{n+1} = f$$

so we recover the original eq if $\alpha = 1$

"The second eq (mass continuity) is intrinsicly considered in equation 2, so does not affect the α parameter selection".

E2) Source of the error:

If we ignore the space and time discretization error, the error of the scheme is located in the evaluation of the velocity in the intermediate step, when $\hat{u}^{n+1}$.

7. ALE Formulations

E1) $\gamma(x,y,z,t) = [2x, ye^t, z]$ Spatial description of a property.

$$\left.\begin{aligned} x &= Xe^t \\ y &= Y + e^t - 1 \\ z &= Z \end{aligned}\right\} \text{equations of movement}$$

$$\left.\begin{aligned} x_m &= \chi + \alpha t \\ y_m &= \mathcal{Y} + \beta t \\ z_m &= \mathcal{Z} \end{aligned}\right\} \text{equations of movement of the mesh}$$

a) Description of the property in terms of the ALE coordinates $(\chi, \mathcal{Y}, \mathcal{Z})$

$$\varphi(X,Y,Z) = [Xe^t, Y + e^t - 1, Z]$$

$$\phi(\chi, \mathcal{Y}, \mathcal{Z}) = [\chi + \alpha t, \mathcal{Y} - \beta t, \mathcal{Z}]$$

$$\gamma(x,y,z,t) = \gamma(\phi(\chi,\mathcal{Y},\mathcal{Z},t),t) = [2(\chi + \alpha t, (\mathcal{Y} - \beta t)e^t, \mathcal{Z}]$$

$$\boxed{\gamma(\chi,\mathcal{Y},\mathcal{Z},t) = [2(\chi + \alpha t), (\mathcal{Y} - \beta t)e^t, \mathcal{Z}]}$$

b) Velocity of the particles?, mesh velocity?

$$\boxed{\begin{aligned} v &= \frac{d}{dt}\varphi(X,Y,Z,t) = [Xe^t, e^t, 0] \\ v_{mesh} &= \frac{d}{dt}\phi(\chi,\mathcal{Y},\mathcal{Z},t) = [\alpha, -\beta, 0] \end{aligned}}$$

c) ALE description of the material temporal derivative of γ.

Mat. Temp. deriv: $\frac{d}{dt}\gamma(\mathcal{X}(\underline{X},t),t)_{ALE} = \frac{\partial \gamma_{ALE}(\underline{\mathcal{X}},t)}{\partial t} + (v - v_{mesh}) \cdot$

$\cdot \nabla(\underline{\mathcal{X}},t)$

$$\frac{\partial \gamma_{ALE}(\underline{\mathcal{X}},t)}{\partial t} = \frac{\partial}{\partial t}\underbrace{\gamma_{ALE}(\emptyset(\mathcal{X},y,z,t),t)} = [2\alpha, e^t(y-\beta(1+t)), 0]$$

$$\gamma_{ALE}(\mathcal{X},y,z,t) = [2(\mathcal{X}+\alpha t), (y-\beta t)e^t, z]$$

$$\begin{cases} \to \frac{d(2(\mathcal{X}+\alpha t))}{dt} = 2\alpha \\ \to \frac{d(e^t(y-\beta t))}{dt} = e^t y - \beta e^t(1+t) = e^t(y-\beta(1+t)) \\ \to \frac{d(z)}{dt} = 0 \end{cases}$$

$$\nabla\gamma(\mathcal{X},t) = \begin{bmatrix} \frac{\partial\gamma_x}{\partial \mathcal{X}} & \frac{\partial\gamma_x}{\partial y} & \frac{\partial\gamma_x}{\partial z} \\ \frac{\partial\gamma_y}{\partial} & \frac{\partial\gamma_y}{\partial y} & \frac{\partial\gamma_y}{\partial z} \\ \frac{\partial\gamma_z}{\partial} & \frac{\partial\gamma_z}{\partial y} & \frac{\partial\gamma_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(v - v_{mesh})$, We need to express v in ALE coordinates!

$v_{mesh} = [\alpha, -\beta, 0]$

$v(\underline{X},t) = [Xe^t, e^t, 0]$; $v(\underline{x},t) = [x, e^t, 0]$; $\left(X = \frac{x}{e^t}\right)$

$v(x,t) = [x, e^t, 0]$, if $x = \mathcal{X} + \alpha t \to v(\underline{\mathcal{X}},t) = [\mathcal{X}+\alpha t, e^t, 0]$

$(v - v_{mesh}) = [\mathcal{X} + \alpha(t-1), e^t + \beta, 0]$

$(v - v_{mesh}) \cdot \nabla\gamma(\underline{\mathcal{X}},t) = [2(\mathcal{X}+\alpha(t-1)), e^t(e^t+\beta), 0]$

$\frac{\partial \gamma_{ALE}(\underline{\mathcal{X}},t)}{\partial t} + (v - v_{mesh}) \cdot \nabla\gamma(\underline{\mathcal{X}},t) = [2(\mathcal{X}+\alpha t), e^t(y - \beta t + e^t), 0]$

So

$$\boxed{\frac{d}{dt}\, \gamma_{ALE}(\chi(\underline{x},t),t) = \left[2(\chi+\alpha t), e^{t}(y-\beta t+e^{t}), 0\right]}$$

E2)

ALE form of the incompressible Navier-Stokes eq's

$$\left.\begin{aligned} \partial_t \underline{u} + (\underline{u} - u_{mesh})\cdot\underline{\nabla}\underline{u} - \nu\underline{\Delta}\underline{u} + \underline{\nabla}p = \underline{f} \\ \underline{\nabla}\cdot\underline{u} = 0 \end{aligned}\right\}$$

Temporal discret. can be done with any time discretization scheme, compute the $U^{n,n+1,n+\dots}$ terms at the deformed mesh points. the spatial derivatives are computed in spatial coordinates!

E3) The mesh update strategies can be briefly subdivided in 2 types:

$$\begin{cases} \text{- mesh regularization} \\ \text{- mesh adaptation} \end{cases}$$

· Mesh regularization

Keep the mesh as regular as possible for avoid excessive distorsion in the elements when they are deforming. When the solid boundary movement of the FSI problem is known, we can apply the next interpolation techniques in the zone of the mesh to modified:

- Transfinite mapping methods:

  Explicit method, low computational cost ✓.

  Imposes restrictions in the mesh topology. X.

- Laplacian smoothing and variational methods:

  We have to solve the Laplace eq. for each component of the node velocity or position. X.

  Is not good for depend what geometries. X.

- Mesh - smoothing and simple interpolation

Simple and general ✓.

Mesh adaptation

We want to improve the accuracy for low computing cost. Move
nodes towards zones with strong solution gradient. For distributing.
the error in the whole domain we need an error estimator. (elliptic or
parabolic diff. equations).

8. Fluid structure interaction

E1) Mass effect problem

The added mass effect appears in incompressible flow problems with
partitioned approaches. In this, the inertia of the fluid in movement
is turned to be a virtual extra mass in the solid. In most of the
cases this extra mass is a second order tensor which relates the accele-
ration in the fluid with the flux applied in the solid boundary. If
the acceleration of the fluid depends of its density, the added
mass effect is negligible when $\rho s >> \rho f$. So, contrary when we have
$\rho s \approx \rho f$ it may either induce large error. or compromise the
convergence.

Main methods for this are:

- Aitken relaxation iterative scheme → For approach the next one
step we have to use information of the 2 previous steps.
- Steepest descent methods.
- Robin-Robin boundary conditions at the fluid-structure interface.

E2)

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f \rightsquigarrow \underline{\underline{\Pi}}\,\underline{u}^{n+1} + \underline{\underline{K}}\,\underline{u}^{n+1} = \underline{f} + \underline{\underline{\Pi}}\,\underline{u}^{n} \,; \quad \delta t = k = 1$$

If we solve for fixed time step $(n+1)$

$U^{i-2}_{\Gamma,2}$

$U^{i-1}_{\Gamma,1}$ $U^{i-1}_{\Gamma,2}$

First Aitken iteration

$\underline{\underline{\Omega_1}}$: $(\underline{\underline{\Pi}} + \underline{\underline{K}})\underline{u}_1^{n+1,i} = \underline{f}_1 + \underline{\underline{\Pi}}\,\underline{u}_1^{n}$ in Ω_1

$k\dfrac{dU^{n+1,i}_{\Gamma,1}}{dx} = -k\dfrac{dU^{n+1,i-1}_{\Gamma,2}}{dx}$ on Γ $\longrightarrow$ we take the flux of the previous iteration, without relaxation.

$u_1 = 0$ on $x=0$

when we have solved this problem:

$$U^{n+1,i}_{\Gamma,1} = U^{n}_{\Gamma,1} + \omega\,(\underbrace{U^{n+1,i^*}_{\Gamma,1}} - U^{n+1,i-1}_{\Gamma,1})$$

value of the bandary with D-N problem (without relaxation)

$$\omega = \frac{U^{i-2}_{\Gamma,1} - U^{i-1}_{\Gamma,1}}{U^{i-2}_{\Gamma,1} - U^{i-1}_{\Gamma,1} + U^{i^*}_{\Gamma,1} - U^{i-1^*}_{\Gamma,1}}$$

$\underline{\underline{\Omega_2}}$: $(\underline{\underline{\Pi}} + \underline{\underline{K}})\underline{U}_2^{n+1,i} = \underline{f}_2 + \underline{\underline{\Pi}}\,\underline{u}_2^{n}$ in Ω_2

$U^{n+1,i}_{\Gamma,2} = \underline{U^{n+1,i}_{\Gamma,1}}$ on Γ

$u_2 = 0$ on ℓ

value with relaxation.

then, when we have solved this, we have to obtain the fluxes at the interface $\rightarrow$ postprocess; $\dfrac{k\,U^{n+1,i}_{\Gamma,2}}{dx}$

Second Aitken iteration

$\underline{\underline{\Omega_1}}$: $(\underline{\underline{\Pi}} + \underline{\underline{K}})\underline{u}_1^{n+1,i+1} = \underline{f}_1 + \underline{\underline{\Pi}}\,\underline{u}_1^{n}$ in Ω_1

$k\,d\,U^{n+1,i+1}_{\Gamma,1} = -k\dfrac{U^{n+1,i}_{\Gamma,2}}{dx}$ on Γ

$u_1 = 0$ on $x=0$

$$u_{\Gamma,1}^{n+1,i+1} = u_{\Gamma,1}^{n} + \omega \left(u_{\Gamma,1}^{n+1,i+1*} - u_{\Gamma,1}^{n+1,i} \right)$$

$$\omega = \frac{u_{\Gamma,1}^{i-1} - u_{\Gamma,1}^{i}}{u_{\Gamma,1}^{i-1} - u_{\Gamma,1}^{i} + u_{\Gamma,1}^{i+1*} - u_{\Gamma,1}^{i*}}$$

Ω_2: $(\underline{\underline{M}} + \underline{\underline{K}}) \underline{u_2}^{n+1,i+1} = \underline{f_2} + \underline{\underline{M}}\, \underline{u_2}^{n}$ in Ω_2

$u_{\Gamma,2}^{n+1,i+1} = u_{\Gamma,1}^{n+1,i+1}$ on Γ

$u_2 = 0$ on $x = l$

When we solved this, then we can obtain the fluxes at the interface as a postprocess $k \dfrac{u_{\Gamma,2}^{n+1,i+1}}{dx}$

E3) Monolithic BDF1, FE discretization, $h = \frac{1}{4}$

Impose BC via Lagrange multipliers, Dirichlet BC at ($x=0$, $x=1$).

The system that we have to solve is:

$$(\underline{\underline{M}} + \underline{\underline{K}}) \underline{u}^{n+1} = \underline{f} + \underline{\underline{M}}\, \underline{u}^{n} \quad ; \quad \delta t = 1$$

$$M_{ij} = \int_{l^{(e)}} N_i N_j \qquad K_{ij} = \int_{l^{(e)}} \cancel{k}^{\,1} \frac{dN_i}{dx} \frac{dN_j}{dx} \quad ; \quad \underline{f} = \int_{l^{(e)}} N_i \cancel{f}^{\,1}$$

$$\underline{\underline{M}}^{①} = \underline{\underline{M}}^{②} = \underline{\underline{M}}^{③} = \underline{\underline{M}}^{④} = \begin{bmatrix} \int_0^{0'25} \left(\frac{0'25-x}{0'25}\right)\left(\frac{0'25-x}{0'25}\right) & \int_0^{0'25} \left(\frac{0'25-x}{0'25}\right)\left(\frac{x}{0'25}\right) \\ \int_0^{0'25} \left(\frac{x}{0'25}\right)\left(\frac{0'25-x}{0'25}\right) & \int_0^{0'25} \left(\frac{x}{0'25}\right)\left(\frac{x}{0'25}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 1/24 \\ 1/24 & 1/2 \end{bmatrix}$$

$$\underline{\underline{K}}^{①} = \underline{\underline{K}}^{②} = \underline{\underline{K}}^{③} = \underline{\underline{K}}^{④} = \begin{bmatrix} \int_0^{0'25} (-1/0'25)(-1/0'25) & \int_0^{0'25} (-1/0'25)(1/0'25) \\ \int_0^{0'25} (1/0'25)(-1/0'25) & \int_0^{0'25} (1/0'25)(1/0'25) \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$\underline{f}^{①} = \underline{f}^{②} = \underline{f}^{③} = \underline{f}^{④} = \begin{bmatrix} 1/8 \\ 1/8 \end{bmatrix}$$

And the assembled matrices are:

$$\underline{\underline{M}} = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix} ; \quad K = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix}$$

Global system (imposing lagrange multipliers)

$$\begin{bmatrix} 49/2 & -95/24 & 0 & 0 & 0 \\ -95/24 & 49/6 & -95/24 & 0 & 0 \\ 0 & -95/24 & 49/6 & -95/24 & 0 \\ 0 & 0 & -95/24 & 49/6 & -95/24 \\ 0 & 0 & 0 & -95/24 & 49/12 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} =$$

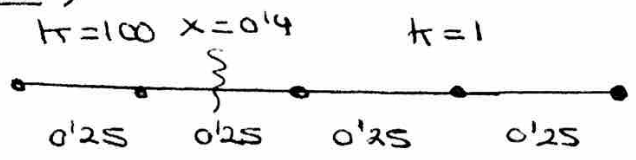
$$= \begin{array}{r} \\ \\ \\ \\ \\ \text{in } x=0 \leftarrow \\ x=l \leftarrow \end{array} \begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 & 0 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 0 & 0 & 1/24 & 1/12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \\ u_5^n \\ 0 \\ 0 \end{bmatrix}$$

The condition number of the final matrix can be obtained as:

$$K = \|A\| \|A^{-1}\|$$ and if we done it with matlab we get:

$$\|A\| = 14'5671 \; ; \; \|A^{-1}\| = 2'6303 \; , \text{ so } \boxed{K = 38'3158}$$

E4)



The matrix is the same:

$$\underline{\underline{M}} = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix}$$

It happens the same with f

But with K, we have to recompute:

$$\underline{\underline{K}}^{(1)} = 100 \cdot \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix}$$

$$\underline{\underline{K}}^{(2)} = \int_{0'25}^{0'4} 100 \cdot \begin{bmatrix} -1/0'25 \cdot -1/0'25 & -1/0'25 \cdot 1/0'25 \\ 1/0'25 \cdot -1/0'25 & 1/0'25 \cdot 1/0'25 \end{bmatrix} + \int_{0'4}^{0'5} 1 \cdot \begin{bmatrix} -1/0'25 \cdot -1/0'25 & \frac{-1}{0'25} \cdot \frac{1}{0'25} \\ 1/0'25 \cdot -1/0'25 & \frac{1}{0'25} \cdot \frac{1}{0'25} \end{bmatrix} =$$

$$= \begin{bmatrix} 241'6 & -241'6 \\ -241'6 & 241'6 \end{bmatrix} \; ; \quad \underline{\underline{K}}^{(3)} = \underline{\underline{K}}^{(4)} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$\underline{\underline{K}} = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 641'6 & -241'6 & 0 & 0 \\ 0 & -241'6 & 245'6 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \; ; \; \underline{\underline{M}}\, \underline{U}^{n+1} + \underline{\underline{K}}\, \underline{U}^{n+1} = \underline{f} + \underline{\underline{M}}\, \underline{U}^{n}$$

we have:

$$\begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} \overset{=0}{\cancel{U_1^{n+1}}} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ \underset{=0}{\cancel{U_5^{n+1}}} \end{bmatrix} + \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 641'6 & -241'6 & 0 & 0 \\ 0 & -241'6 & 245'6 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \cdot$$

$+$

$$\begin{bmatrix} \overset{=0}{\cancel{u_1^{n+1}}} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \underset{\parallel \atop 0}{\cancel{u_5^{n+1}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} \end{bmatrix} + \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} \overset{=0}{\cancel{u_1^{n}}} \\ u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ \underset{\parallel \atop 0}{\cancel{u_5^{n}}} \end{bmatrix}$$