Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Computational Solid Mechanics

Assignment 3.1

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1.1 Problem statement

A two-dimensional solid is contained in the $\{X1, X2\}$ coordinate plane relative to an orthonormal cartesian basis E_I, $I = 1, 2, 3$. The solid is initially square in shape and is enclosed in a rigid truss frame hinged at the corners A, B, C, and D of the square, so that the sides AB, BC, CD and DA cannot change their length. The deformation is presumed homogeneous and is parametrized by the angle α rotated by the sides DA and BC.

1. Write the deformation mapping in terms of α
2. Compute the deformation gradient $\boldsymbol{F}$ and the right Cauchy-Green deformation tensor $\boldsymbol{C}$.
3. Compute and plot the variation in volume of the solid as a function of α.
4. At what point do the deformations cease to be admissible? Interpret geometrically.
5. Compute the change in length of the diagonals AC and BD, and the change in the angle β subtended by them. Interpret geometrically. Plot the change of lengths and the change of angle β as a function of α.

1.2 Solution

I wrote a Matlab script to solve the problem:

```
a = sym('a', 'positive'); A = sym('A', 'positive');
b = sym('b', 'positive'); B = sym('B', 'positive');
alpha = sym('alpha', 'real');

X = [1 0
     0 1];

x = [1 sin(alpha);
     0 cos(alpha)];
% x = FX
F = x / X;

C = simplify(F'*F);
J = det(F);
critical = solve(J==0);

T1 = [1 1]';
T2 = [1 -1]';

T1 = T1/norm(T1);
T2 = T2/norm(T2);

lambda1 = simplify(sqrt(T1'*C*T1));
lambda2 = simplify(sqrt(T2'*C*T2));
beta = acos(T1' * C * T2);

% Not appended: printing of results and plotting
```

The output of the program was:

```
>> exercise2
Deformation gradient F
[ 1, sin(alpha)]
[ 0, cos(alpha)]

Cauchy-Green:
[          1, sin(alpha)]
[ sin(alpha),          1]

J = cos(alpha)
Critical alpha = pi/2
lambda1 = (sin(alpha) + 1)^(1/2)
lambda2 = (1 - sin(alpha))^(1/2)
beta/beta0 = pi/2
```

Details:

The jacobian becomes negative for $\alpha = \pi/2$ because the geometry folds onto a line.

The changes in length of the diagonals are expressed as in ratios $\lambda_i = l_i/L_i$. Diagonal AC corresponds to λ_1 and BD corresponds to λ_2.

Beta is constant since it only depends on the length of the sides, which also remain unchanged.

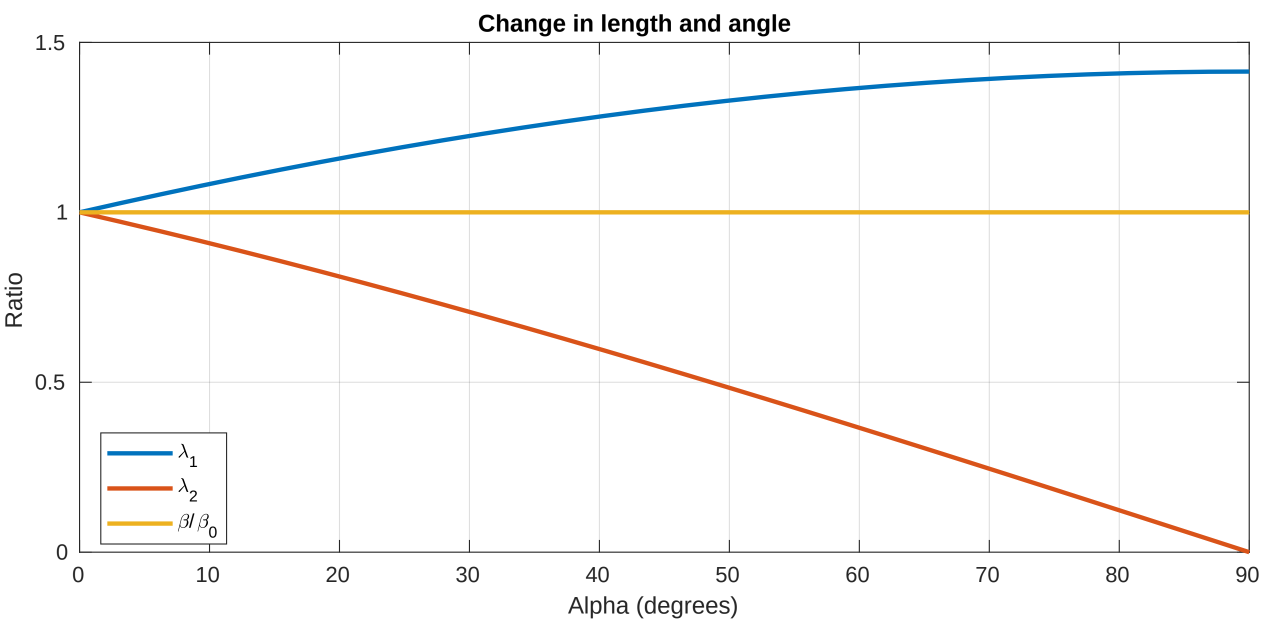


Figure 1: Relationship of length and angle ratios with alpha

2 Exercise 3

2.1 Problem statement

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame $\{X^1, X^2, X^3\}$, whose axis aligned with the X^3 direction. Its normal cross section occupies a region Ω in the $\{X^1, X^2\}$ plane of boundary $\partial\Omega$. An anti-plane shear deformation of the solid can be defined as one for which the deformation mapping is of the form:

$$\varphi_1 = X^1 \qquad \varphi_2 = X^2 \qquad \varphi_3 = X^3 + w(X^1, X^2) \tag{1}$$

The spatial and material reference frames are taken to coincide, and the function w is defined over Ω.

1. Sketch the deformation of the region Ω.
   - Compute the deformation gradient field $\boldsymbol{F}$, the right Cauchy-Green deformation tensor $\boldsymbol{C}$, and the Jacobian J of the deformation field in terms of w.
   - Does the solid change volume during the deformation?
   - Are the local impenetrability conditions satisfied?

2. Consider the unit vectors:

$$\boldsymbol{A} = \frac{w_{,1}\boldsymbol{E}_1 + w_{,2}\boldsymbol{E}_2}{\sqrt{w_{,1}^2 + w_{,2}^2}} \qquad \boldsymbol{B} = \frac{-w_{,2}\boldsymbol{E}_1 + w_{,1}\boldsymbol{E}_2}{\sqrt{w_{,1}^2 + w_{,2}^2}} \tag{2}$$

   where $\{E_I\}$ $I = 1, 2, 3$ are the (orthonormal) material basis vectors.
   - How are $\boldsymbol{A}$ and $\boldsymbol{B}$ related to the level contours of $w(X^1, X^22)$?
   - Compute (in terms of w) the change in length (measured by the corresponding stretch ratios) of $\boldsymbol{A}$ and $\boldsymbol{B}$, as well as the change in the angle subtended by $\boldsymbol{A}$ and $\boldsymbol{B}$.
   - Interpret the results.

3. Using the Piola transformation, compute (in terms of w) the change in area of, and in the normal to, an infinitesimal material area contained in the $\{X^1, X^2\}$ plane.

4. Derive an integral expression for the deformed area of the domain Ω.

5. Let the boundary $\partial\Omega$ of Ω be defined parametrically by the equations

$$X^1 = X^1(S) \qquad X^2 = X^2(S) \tag{3}$$

   where $0 \leq S \leq L$ is the arc-length measured along $\partial\Omega$. Note that $\boldsymbol{E}_1 X^1(S)/dS + \boldsymbol{E}_2 X^2(S)/dS$ is the unit vector tangent to $\partial\Omega$. Derive an integral expression for the perimeter of the deformed boundary $\varphi(\partial\Omega)$.

2.2 Solution

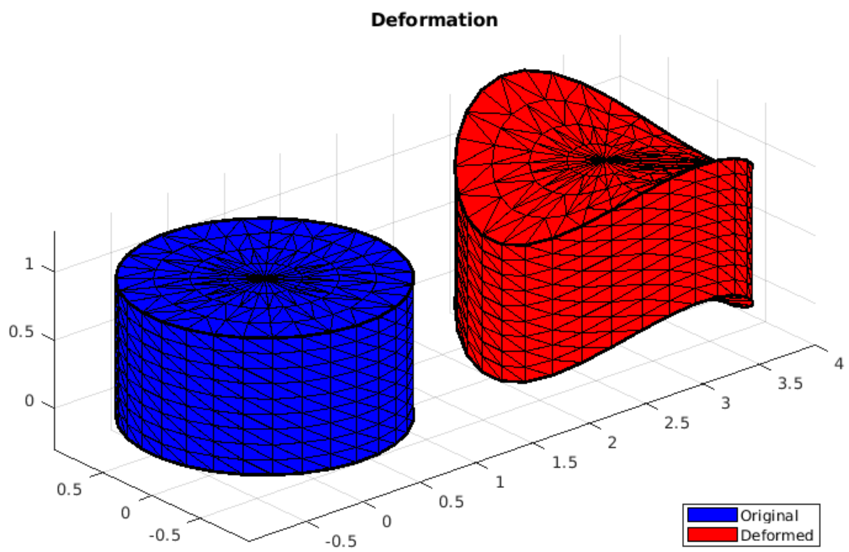
First off we calculate the defformation gradient and Cauchy-Green tensors:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_{,1} & w_{,2} & 1 \end{bmatrix} \qquad C = \begin{bmatrix} w_{,1}^2 + 1 & w_{,1}w_{,2} & w_{,1} \\ w_{,1}w_{,2} & w_{,2}^2 + 1 & w_{,2} \\ w_{,1} & w_{,2} & 1 \end{bmatrix} \tag{4}$$

The jacobian becomes:

$$J = \det F = 1 \tag{5}$$

The solid does therefore not to change volume during deformation. Local impenetrability is satisfied since $J \geq 0, \forall \boldsymbol{X} \in \Omega$. Figure 2 shows a sample deformation.



$$w(X, Y) = -\frac{XY}{3}$$

Figure 2: Example deformation. Note that the deformed domain has been translated $f : \boldsymbol{X} \mapsto \boldsymbol{X} + 3\boldsymbol{E}_1$ for clarity.

The unit vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ relate to the contours of w by being normal($\boldsymbol{A}$) and tangent($\boldsymbol{B}$) to them. We see this since:

$$\boldsymbol{A} = \frac{\nabla w}{||\nabla w||} \qquad \left.\begin{matrix} \boldsymbol{B} \cdot \boldsymbol{A} = 0 \\ \boldsymbol{B} \cdot \boldsymbol{E}_3 = 0 \end{matrix}\right\} \tag{6}$$

The strech ratios are easy enough to calculate:

$$\lambda_A = \boldsymbol{A} \cdot \boldsymbol{CA} = w_{,1}^2 + w_{,2}^2 + 1 \qquad \lambda_B = \boldsymbol{B} \cdot \boldsymbol{CB} = 1 \tag{7}$$

Their angle remains 90 degrees apart:

$$\beta = \arccos(\boldsymbol{A} \cdot \boldsymbol{CB}) = \frac{\pi}{2} \tag{8}$$

This implies that the shape of the contours, when projected to the 2D plane, stays the same after transformation.

Moving on to the change in area, it can be calculated as:

$$\frac{da}{dA} = ||J\boldsymbol{F}^T\boldsymbol{N}|| = \sqrt{w_{,1}^2 + w_{,2}^2 + 1} \tag{9}$$

The change in angle is

$$\cos\gamma = ||\boldsymbol{N}\cdot\boldsymbol{n}|| = \frac{||\boldsymbol{N}\cdot J\boldsymbol{F}^T\boldsymbol{N}||}{||J\boldsymbol{F}^T\boldsymbol{N}||} = \frac{1}{\sqrt{w_{,1}^2 + w_{,2}^2 + 1}} \tag{10}$$

If we want to integrate the area we could do:

$$a = \int_\omega da = \int_{X_1^2+X_2^2\leq R^2} (\frac{da}{dA})^{-1} da = \int_{X_1^2+X_2^2\leq R^2} \frac{1}{\sqrt{w_{,1}^2 + w_{,2}^2 + 1}} dA \tag{11}$$

at this point cylindrical coordinates would become useful:

$$\left.\begin{array}{c} X_1 = P\cos(\Theta) \\ X_2 = P\sin(\Theta) \\ X_3 = Z \end{array}\right\} \qquad \left.\begin{array}{c} x_1 = \rho\cos(\theta) \\ x_2 = \rho\sin(\theta) \\ x_3 = z \end{array}\right\} \tag{12}$$

Then the area becomes:

$$a = \int_0^R \int_0^{2\pi} \frac{d\Theta\, dP}{\sqrt{w_{,1}^2 + w_{,2}^2 + 1}} \tag{13}$$

Calculating the perimeter is much of the same. Let's name T the tangent unit vector:

$$\boldsymbol{T} = \frac{\boldsymbol{E}_1 X^1(S) + \boldsymbol{E}_2 X^2(S)}{||\boldsymbol{E}_1 X^1(S) + \boldsymbol{E}_2 X^2(S)||} \tag{14}$$

Then the perimeter can be calculated as

$$\Pi = \int_{\partial\omega} ds = \int_{\partial\Omega} \lambda_T dS = \int_{\partial\Omega} ||\boldsymbol{T}\cdot\boldsymbol{C}\boldsymbol{T}||\, dS = \int_0^L ||\boldsymbol{T}\cdot\boldsymbol{C}\boldsymbol{T}||\, dS \tag{15}$$