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Computational Mechanics Master Program

Computational Solid Mechanics

Solution of Homework 1_b

Consider the homogeneous deformation of a hinged rigid truss:

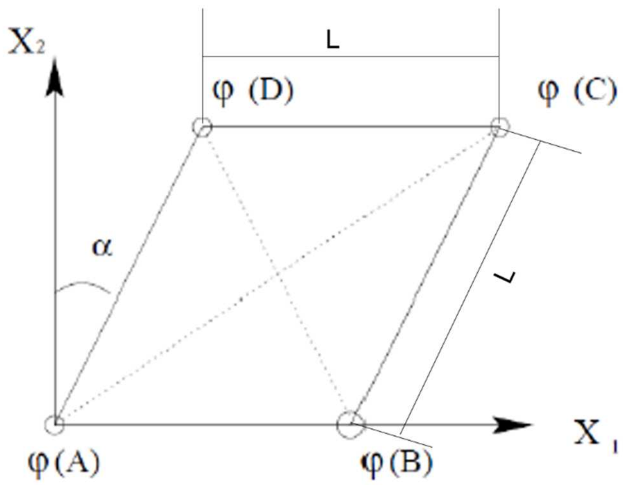

Figure 1. Hinged rigid truss considered for the homework.

For α = 0, the rigid truss encloses a square with side length L. Considering that the deformation map ϕ (**X**) is parametrized by the angle α, find:

<u>1) The deformation map ϕ (**X**) is terms of α:</u>

There is a shift in the X_1 direction and a change in the area within the hinged rigid truss.

$$\phi(\mathbf{X}) = [\ X_1 + \sin(\alpha)X_2 \quad \cos(\alpha)X_2\ ]^T$$

<u>2) The deformation gradient **F** and the right Cauchy-Green deformation tensor **C:**</u>

$$\mathbf{F} = \mathbf{GRAD}(\varphi\ (\mathbf{X}))$$

$$\mathbf{F} = \begin{bmatrix} 1 & \sin(\alpha) \\ 0 & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

$$\mathbf{C} = \begin{bmatrix} 1 & \sin(\alpha) \\ \sin(\alpha) & \sin^2(\alpha) + \cos^2(\alpha) \end{bmatrix}$$

3) The variation in area of the solid as a function of alpha. Plot the variation:

Calculating the area enclosed by the hinged rigid truss (Figure 1):

$$A = 2\left(\frac{L^2\cos(\alpha)\sin(\alpha)}{2}\right) + L\cos(\alpha)(L - L\sin(\alpha))$$

$$A = L^2\cos(\alpha)$$

Considering L = 1 unit of length:

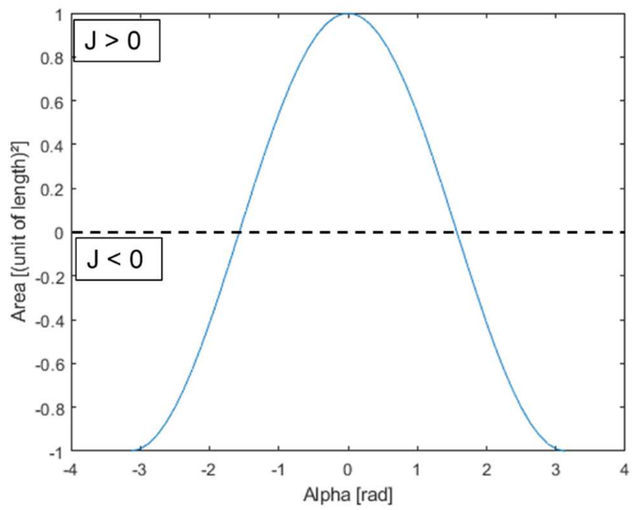

Figure 2. Variation in area of solid as a function of alpha.

According to Figure 2, the area values below the black dashed line are negative. Therefore, for the values of α which result in a negative area, the jacobian J is smaller than zero and such range of α is physically inacceptable.

4) The point which the deformation ceases to be acceptable. Interpret geometrically.

For a deformation to be acceptable, the jacobian J must be positive (J>0)

$$J^2 = \det(\mathbf{C})$$

$$J^2 = \sin^2(\alpha) + \cos^2(\alpha) - \sin^2(\alpha)$$

$$J = \cos(\alpha) > 0 \therefore \frac{-\pi}{2} < \alpha < \frac{\pi}{2}$$

If the jacobian J is negative, the area of the solid would be negative, which is physically impossible. In such case, the negative area means that parts of the structure would collapse, and the structure would turn “inside out”.

5) Compute the change in length of the internal diagonals and the subtended angle β in terms of alpha. Plot their variations in terms of alpha. Interpret geometrically.

Considering that the edges of the truss structure have constant size L (Figure 1), the length of diagonal AC as a function of the angle α was calculated as the hypotenuse of the following triangle (highlighted in thick black lines):

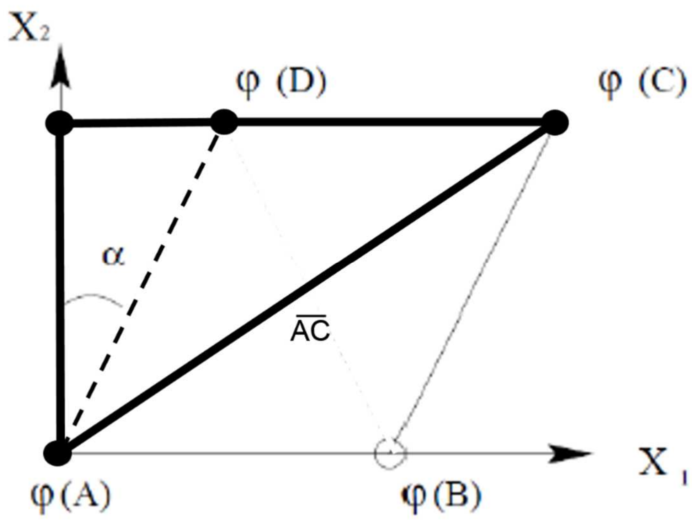

Figure 1. Triangle (in thick black lines) considered to calculate the length of diagonal AC as a function of angle α.

Following the same idea, the length of the diagonal BD was calculated considering the following triangle (in black thick lines):

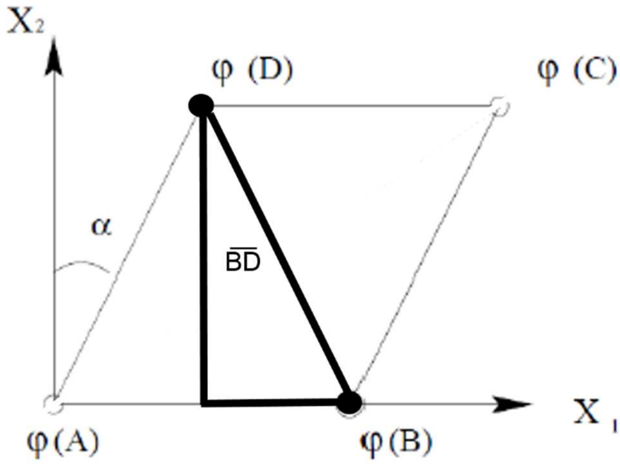

Figure 2. Triangle (highlighted in thick black lines) considered to calculate the length of diagonal BD as a function of angle α.

Considering the triangles highlighted in Figures 3 and 4, the length of the diagonals AC and BD are the defined as follows:

$$\overline{AC} = L\sqrt{2(1 + \sin(\alpha))}$$

$$\overline{BD} = L\sqrt{2(1 - \sin(\alpha))}$$

Since the hinged truss will always have a rhombic shape, the angle subtended by the internal diagonals will remain π/2. Therefore, the angle β is independent of the angle α and will always have the value of π/2.

$$\beta = \frac{\pi}{2}$$

For the plot, it was considered L =1.

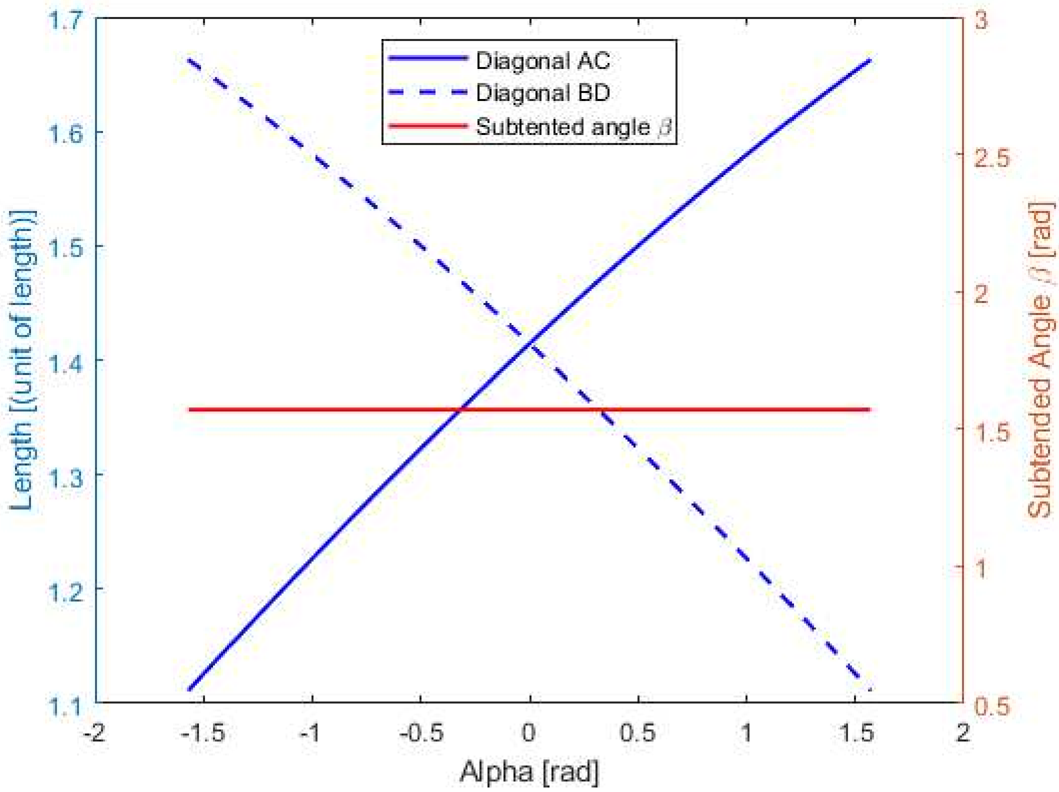

Figure 3. Variation of length of diagonals AC and BD (in blue) and subtended angle β (in red) for acceptable values of alpha.

According to Figure 5, if the angle alpha → π/2, rotating the structure to the right, the length of the diagonal AC tends to the value 2L and the length of the diagonal BD tends to zero. The same opposite behavior is obtained if the angle alpha → - π/2. Both behaviors are in agreement, since as alpha approaches either π/2 or -π/2, the structure tends to a linear shape with length 2L. If the angle alpha → - π/2, the linear shape coincides with the diagonal BD. If the angle alpha → π/2, the linear shape coincides with the diagonal AC.

Solution of Homework 1_a

Consider the following deformation of a rubber matrix (in green) with a reinforced wire grid, which is inextensible:

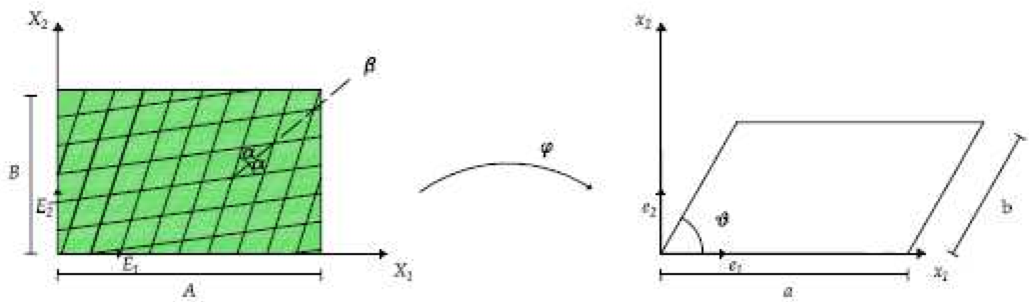

Figure 4. Deformation of rubber block with a reinforced wire grid.

The wire grid is inextensible and rigidly attached to the rubber matrix. Therefore, the wire grid follows the deformation of the rubber matrix. The angle β is the bisection between the grid wires, defining the angle α, and it is measured from the axis X1.

<u>1) Define the right Cauchy-Green deformation tensor C as a function of stretches a/B, b/B and angle θ.</u>

To obtain the stretch in a certain direction, we can use the following equation:

$$\lambda^2 = \mathbf{NCN} \tag{1}$$

For the stretch in X1 direction:

$$\left(\frac{a}{A}\right)^2 = [1\ 0]\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Yielding:

$$\left(\frac{a}{A}\right)^2 = C_{11}$$

For the stretch in X2 direction:

$$\left(\frac{b}{B}\right)^2 = [0\ 1]\begin{bmatrix} \left(\frac{a}{A}\right)^2 & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Yielding:

$$\left(\frac{b}{B}\right)^2 = C_{22}$$

Since the right Cauchy-Green tensor C is symmetric, $C_{12} = C_{21} = C$. Therefore, we can use the following relationship to define the term C:

$$\cos(\theta) = \lambda_{X1}^{-1}\lambda_{X2}^{-1}\,\mathbf{N}_{X1}\,\mathbf{C}\,\mathbf{N}_{X2} \tag{2}$$

Replacing the known values in the Equation 2, the following is obtained:

$$\cos(\theta) = \frac{A}{a}\frac{B}{b}[1\ 0]\begin{bmatrix} \left(\frac{a}{A}\right)^2 & C \\ C & \left(\frac{b}{B}\right)^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for C**:**

$$C = C_{12} = C_{21} = \frac{ab}{AB}\cos(\theta)$$

Writing the tensor C with its components:

$$\mathbf{C} = \begin{bmatrix} \left(\frac{a}{A}\right)^2 & \frac{ab}{AB}\cos(\theta) \\ \frac{ab}{AB}\cos(\theta) & \left(\frac{b}{B}\right)^2 \end{bmatrix}$$

2) For what values of β the sides of rubber matrix remain at 90◦?

In this case, since the sides of the block remain at 90_degrees, the components C_{21} and C_{12} are equal to zero and the components C_{11} and C_{22} remain the same. Also, since the reinforced grid is inextensible, the stretch λ in the direction of the wires is 1. Therefore, we can find a relationship for β through the following Equation 1. Considering the two directions of the wires, we can define the normal vector N in each direction:

$$\mathbf{N}_1 = [\cos(\beta - \alpha) \quad \sin(\beta - \alpha)]^T$$

$$\mathbf{N}_2 = [\cos(\beta + \alpha) \quad \sin(\beta + \alpha)]^T$$

Replacing both normal vectors in Equation 1, and considering that λ = 1 in both directions and $C_{12} = C_{21} = 0$, the following system of equations is obtained:

$$1 = \left(\frac{a}{A}\right)^2 \cos^2(\beta - \alpha) + \left(\frac{b}{B}\right)^2 \sin^2(\beta - \alpha)$$

$$1 = \left(\frac{a}{A}\right)^2 \cos^2(\beta + \alpha) + \left(\frac{b}{B}\right)^2 \sin^2(\beta + \alpha)$$

After applying trigonometric identities and further manipulations, the following relationship for values of β to guarantee θ = 90° is obtained:

$$\beta = \frac{n\pi}{2}, \quad \text{for n } = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots$$

It is important to mention that I could not manipulate the system of equations to arrive in such relationship for β.

3) Compute the stretch ratios a/A and b/B as a function of the angles α, β and θ.

Considering the right Cauchy-Green tensor **C** defined in exercise 1:

$$\boldsymbol{C} = \begin{bmatrix} \left(\frac{a}{A}\right)^2 & \frac{ab}{AB}\cos(\theta) \\ \frac{ab}{AB}\cos(\theta) & \left(\frac{b}{B}\right)^2 \end{bmatrix}$$

Considering the normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ in the direction the wires:

$$\boldsymbol{N}_1 = [\cos(\beta - \alpha) \quad \sin(\beta - \alpha)]^T$$

$$\boldsymbol{N}_2 = [\cos(\beta + \alpha) \quad \sin(\beta + \alpha)]^T$$

And considering that the stretches along the wires are equal to 1, we can employ Equation 1 for directions $\mathbf{N}_1$ and $\mathbf{N}_2$ to obtain the following nonlinear system of equations:

$$1 = \left(\frac{a}{A}\right)^2 \cos^2(\beta + \alpha) + \left(\frac{b}{B}\right)^2 \sin^2(\beta + \alpha) + 2\cos(\theta)\left(\frac{a}{A}\right)\left(\frac{b}{B}\right)\cos(\beta + \alpha)\sin(\beta + \alpha)$$

$$1 = \left(\frac{a}{A}\right)^2 \cos^2(\beta - \alpha) + \left(\frac{b}{B}\right)^2 \sin^2(\beta - \alpha) + 2\cos(\theta)\left(\frac{a}{A}\right)\left(\frac{b}{B}\right)\cos(\beta - \alpha)\sin(\beta - \alpha)$$

After manipulations and employing trigonometric identities, it is possible to describe a/A and b/B ratios in terms of α, β and θ. Unfortunately, I was not able to find such relationship.

4) Considering β = 0, and that the rubber matrix is stretched in X1 direction and compressed in the direction X2, define a nonlinear poisson ratio v in terms of α.

As only stretches (elongation and compression) are applied to the rubber matrix, the terms C_{12} and C_{21} are equal to zero. Therefore, the following relationship can be employed to calculate the Lagrangian strain tensor E:

$$\boldsymbol{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \tag{3}$$

Replacing the known values in Equation (3), the following Lagrangian strain tensor is obtained:

$$\boldsymbol{E} = \frac{1}{2}\begin{bmatrix} \left(\frac{a}{A}\right)^2 - 1 & 0 \\ 0 & \left(\frac{b}{B}\right)^2 - 1 \end{bmatrix}$$

To obtain a relationship between the components of the Lagrangian strain tensor **E** and the angle α, the following relationship can be employed:

$$\lambda^2 = 1 + 2\boldsymbol{NEN} \tag{4}$$

Considering that the stretch is 1 in the direction of the wires and considering the definition of normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$, Equation (4) can be written for both N_1 and N_2. The output is the following system of equations:

$$1 = \cos^2(\beta - \alpha)\left[\left(\frac{a}{A}\right)^2 - 1\right] + \sin^2(\beta - \alpha)\left[\left(\frac{b}{B}\right)^2 - 1\right]$$

$$1 = \cos^2(\beta + \alpha)\left[\left(\frac{a}{A}\right)^2 - 1\right] + \sin^2(\beta + \alpha)\left[\left(\frac{b}{B}\right)^2 - 1\right]$$

After manipulation and employing trigonometric identities, a relationship can be found for poisson v as a function of α. Unfortunately, I was not able to find such relationship.

5) For what values of α, the poisson v = 0.5?

Unfortunately, I was not able to find such the relationship.