UNIVERSITAT POLITÈCNICA DE CATALUNYA, BARCELONA

MSC. COMPUTATIONAL MECHANICS ERASMUS MUNDUS

ASSIGNMENT 1: DIRECT STIFFNESS METHOD

Computational Structural Mechanics & Dynamics

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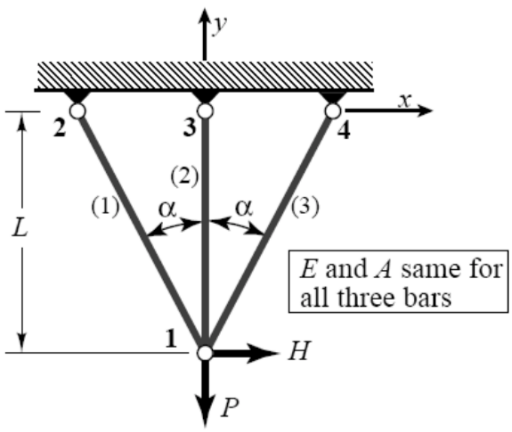
Assignment 1

On "The Direct Stiffness Method":

Consider the truss problem defined in the Figure. All geometric and material properties: L, α, E and A, as well as the applied forces P and H, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.



(a) Show that the master stiffness equations are

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -cs^2 \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which $c = \cos\alpha$ and $s = \sin\alpha$. Explain from physics why the 5th row and column contain only zeros.

Solution: For a given coordinate system, the (4×4) stiffness matrix of a truss element at an angle α, with each node having 2 degrees of freedom is given by:

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Considering the angle measurement in the counter-clockwise direction as a general convention, the three elements in this truss problem form the following angles with the global x-coordinate.

Element	**Angle**	**sin (Angle)**	**cos (Angle)**
1	$\pi/2+\alpha$	cos α	-sin α
2	$\pi/2$	1	0
3	$\pi/2-\alpha$	cos α	sin α

Since we consider c = cos α and s = sin α, we know by trigonometry that the length of element 1 and element 3 are given as, $L^{(1)} = L^{(3)} = L/c$ and $L^{(2)} = L$. Hence with the known material and geometric properties, we write the element stiffness matrices as,

$$\mathbf{K}^{(1)} = \frac{EAc}{L}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}^{(3)} = \frac{EAc}{L}\begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix}$$

Now, to assemble the global stiffness matrix of the system, we need to augment the element stiffness matrices to the same size as the global stiffness matrix. Given that the truss problem has 8 degrees of freedom, we get the following element stiffness matrices,

$$\mathbf{K}^{(1)} = \frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 \\ -c^2s & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}^{(3)} = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ c^2s & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}$$

The augmentation of the element matrices makes the assembly process very easy, as we just need to add the three matrices to get the global stiffness matrix as,

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}$$

The obtained global stiffness matrix is the same as suggested in the problem. Considering the direction of the applied forces, the master stiffness equations are given as,

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

We also notice that the 5th row and column of the matrix contain only zeros. This behaviour can be explained by looking at few features of a truss.

- A truss can only be subjected to axial loading
- The deformation is along the axial direction
- A truss cannot sustain shear and moment
- The load can only be applied at the two ends

Considering these features of a truss element, we see that element 2 is pinned at node 3 and can only deform in the axial direction i.e. along the global vertical direction. This means the contribution of node 3 in the horizontal displacement is zero which makes the corresponding 5th row and column as null. It is interesting to note that this behaviour is only observed for element 2 because it is aligned with the global axis, whereas the other truss elements are inclined to the global axis.

(b) Apply the BCs and show the 2-equation modified stiffness system.

Solution: In the given problem, the truss elements are pinned at nodes 2, 3 and 4, which results in zero displacement in both horizontal and vertical directions. These known displacements $u_{x2} = 0$, $u_{y2} = 0$, $u_{x3} = 0$, $u_{y3} = 0$, $u_{x4} = 0$, $u_{y4} = 0$, are the boundary conditions (BCs) of the problem. On application of these BCs, we can remove the corresponding rows to these displacements from equation 1. The system of equations can be further reduced by removing the corresponding columns since the known displacements are equal to zero. This leaves us with a modified stiffness system with only u_{x1} and u_{y1} as the unknowns given as,

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{2}$$

(c) Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

Solution: Solving the modified stiffness system as given in equation 2, we get,

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \frac{L}{EA}\begin{bmatrix} 1/(2cs^2) & 0 \\ 0 & 1/(1+2c^3) \end{bmatrix}\begin{bmatrix} H \\ -P \end{bmatrix}$$

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} HL/(2EAcs^2) \\ -PL/(EA(1+2c^3)) \end{bmatrix} \tag{3}$$

As $\alpha \to 0$, $c \to 1$ and $s \to 0$. Therefore,

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} HL/0 \\ -PL/(3EA) \end{bmatrix}$$

This makes physical sense since $\alpha \to 0$ means that the three bars are pointing downwards which is equivalent to a single bar with three times the area pinned at one end and vertical load P applied at the free end. This gives us an axial stiffness of $3EA/L$ which is evident in the expression of u_{y1}.

If $H \neq 0$, the solution completely 'blows up' as $u_{x1} \to \infty$. The reduced stiffness matrix becomes singular as the pinned support does not resist the moment created by the horizontal force, allows free rotations and makes it a mechanism giving an undetermined solution. The features of truss element (specified in part (a)) also states that a truss cannot sustain shear or moment and deformation can only take place along the axial direction which gives another reason for the solution to blow up in this case.

Now, as $\alpha \to \pi/2$, $c \to 0$ and $s \to 1$. Therefore,

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} HL/0 \\ -PL/(EA) \end{bmatrix}$$

This is because, with $\alpha \to \pi/2$, the two bars are parallel to the top surface of the structure making the length $L^{(1)} = L^{(3)} = L/c \to \infty$ which corresponds to the case of a single bar pinned at one end and vertical load P applied at the free end. This gives us an axial stiffness of EA/L evident in the expression of u_{y1} and an undefined solution in the horizontal direction.

(d) Recover the axial forces in the three members. Partial answer: $F^{(3)} = -H/(2s)+Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

<u>Solution</u>: To recover the axial forces of the three elements in their local axes, first we need to find the local displacements using the obtained global displacements. This is done by using the rotation matrix $\mathbf{T}$,

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

<u>For element 1:</u>

Global displacements,

$$\mathbf{u}^{(1)} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix}$$

Local displacements are given as,

$$\overline{\mathbf{u}}^{(1)} = \mathbf{T}^{(1)}\mathbf{u}^{(1)}$$

$$\overline{\mathbf{u}}^{(1)} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -su_{x1} + cu_{y1} \\ -cu_{x1} - su_{y1} \\ 0 \\ 0 \end{bmatrix}$$

Now, deformation

$$d^{(1)} = \overline{u}_{x2} - \overline{u}_{x1} = su_{x1} - cu_{y1}$$

Using the results from equation 3, we get,

$$d^{(1)} = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

Using the Force-deformation relationship, we get the axial force as,

$$F^{(1)} = \frac{EA}{L^{(1)}} d^{(1)} = \frac{EAc}{L} \left[\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)} \right]$$

$$\boxed{F^{(1)} = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}} \tag{4}$$

For element 2:

Global displacements,

$$\mathbf{u}^{(2)} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix}$$

Local displacements are given as,

$$\overline{\mathbf{u}}^{(2)} = \mathbf{T}^{(2)} \mathbf{u}^{(2)}$$

$$\overline{\mathbf{u}}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_{y1} \\ -u_{x1} \\ 0 \\ 0 \end{bmatrix}$$

Now, deformation

$$d^{(2)} = \overline{u}_{x3} - \overline{u}_{x1} = -u_{y1}$$

Using the results from equation 3 again, we get,

$$d^{(2)} = \frac{PL}{EA(1+2c^3)}$$

Using the Force-deformation relationship, we get the axial force as,

$$F^{(2)} = \frac{EA}{L^{(2)}} d^{(2)} = \frac{EA}{L}\left[\frac{PL}{EA(1+2c^3)}\right]$$

$$\boxed{F^{(2)} = \frac{P}{(1+2c^3)}} \tag{5}$$

For element 3:

Global displacements,

$$\mathbf{u}^{(3)} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix}$$

Local displacements are given as,

$$\overline{\mathbf{u}}^{(3)} = \mathbf{T}^{(3)} \mathbf{u}^{(3)}$$

$$\overline{\mathbf{u}}^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} su_{x1} + cu_{y1} \\ -cu_{x1} + su_{y1} \\ 0 \\ 0 \end{bmatrix}$$

Now, deformation

$$d^{(3)} = \overline{u}_{x4} - \overline{u}_{x1} = -su_{x1} - cu_{y1}$$

Using the results from equation 3, we get,

$$d^{(3)} = -\frac{HLs}{2EAcs^2} + \frac{PLc}{EA(1+2c^3)}$$

Using the Force-deformation relationship, we get the axial force as,

$$F^{(3)} = \frac{EA}{L^{(3)}} d^{(3)} = \frac{EAc}{L}\left[-\frac{HLs}{2EAcs^2} + \frac{PLc}{EA(1+2c^3)}\right]$$

$$\boxed{F^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}} \tag{6}$$

We notice again, that when $\alpha \to 0$, $c \to 1$ and $s \to 0$. This makes the the axial forces $F^{(1)}$ and $F^{(3)}$ to blow up since the three bars can be effectively considered a single bar with area $3A$ with an axial stiffness of $3EA/L$. It is also seen that if $H = 0$, the total force is divided equally by each bar under tensile load $P/3$. But if $H \neq 0$, both $F^{(1)}$ and $F^{(3)}$ gives undefined solution due to the pinned support at the top not resisting moment and making it a mechanism.

Assignment 2

Dr. Who proposes "improving" the result for the example truss of the 1^{st} lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

Solution: Addition of an extra node at the midpoint of element 3 does not affect the inclination angles of the truss elements and thus the elemental stiffness matrices for element 1 and 2 remains same with increase in the total degrees of freedom to 10. Hence the augmented element stiffness matrices are given as,

$$\mathbf{K}^{(1)} = \frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 & 0 & 0 \\ -c^2s & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Also, the new elements 3 and 4 are half in length of the original element 3 i.e. $L^{(3)} = L^{(4)} = L/(2c)$ resulting in similar stiffness matrices given as,

$$\mathbf{K}^{(3)} = \frac{2EA}{L}\begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s & 0 & 0 \\ c^2s & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s & 0 & 0 \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\mathbf{K}^{(4)} = \frac{2EA}{L}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & cs^2 & c^2s & -cs^2 & -c^2s\\
0 & 0 & 0 & 0 & 0 & 0 & c^2s & c^3 & -c^2s & -c^3\\
0 & 0 & 0 & 0 & 0 & 0 & -cs^2 & -c^2s & cs^2 & c^2s\\
0 & 0 & 0 & 0 & 0 & 0 & -c^2s & -c^3 & c^2s & c^3
\end{bmatrix}
$$

Upon assembly, the global stiffness matrix is given as,

$$
\mathbf{K} = \frac{EA}{L}\begin{bmatrix}
3cs^2 & c^2s & -cs^2 & c^2s & 0 & 0 & -2cs^2 & -2c^2s & 0 & 0\\
c^2s & 3c^3+1 & c^2s & -c^3 & 0 & -1 & -2c^2s & -2c^3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 & 0 & 0\\
c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2cs^2 & -2c^2s-1 & 0 & 0 & 0 & 1 & 4cs^2 & 4c^2s & -2cs^2 & -2c^2s\\
-2c^2s & -2c^3 & 0 & 0 & 0 & 0 & 4c^2s & 4c^3 & -2c^2s & -2c^3\\
0 & 0 & 0 & 0 & 0 & 0 & -2cs^2 & -2c^2s & 2cs^2 & 2c^2s\\
0 & 0 & 0 & 0 & 0 & 0 & -2c^2s & -2c^3 & 2c^2s & 2c^3
\end{bmatrix}
$$

It is interesting to note that Dr. Who's reasoning of adding an extra node doesn't make any difference in the 5th row and column of the global stiffness matrix which are still null as explained earlier. Now, in order to solve the stiffness equations we impose boundary conditions. Since the truss elements are pinned at nodes 2, 3 and 5, we get the reduced stiffness system corresponding to unknowns u_{x1}, u_{y1}, u_{x4} and u_{y4} as,

$$
\frac{EA}{L}\begin{bmatrix}
3cs^2 & c^2s & -2cs^2 & -2c^2s\\
c^2s & 3c^3+1 & -2c^2s & -2c^3\\
-2cs^2 & -2c^2s-1 & 4cs^2 & 4c^2s\\
-2c^2s & -2c^3 & 4c^2s & 4c^3
\end{bmatrix}
\begin{bmatrix} u_{x1}\\ u_{y1}\\ u_{x4}\\ u_{y4}\end{bmatrix}
=
\begin{bmatrix} H\\ -P\\ 0\\ 0\end{bmatrix}
\tag{7}
$$

We observe that the stiffness matrix of the reduced system is singular, therefore the system of equations are not compatible and there is no distinct solution to the problem. Physically, this is because the structure is not fully constrained in space and behaves like a mechanism. This condition known as rigid body motion implies that with little perturbation, the system becomes unstable in nature. Hence, Dr. Who's proposal of inserting an extra node in the middle of an element is ineffective since a truss element can only be subjected to axial loading.