Assignment 401

0.1

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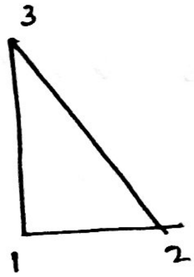
1.) $\varepsilon_r = \frac{\partial u}{\partial r}$; $\varepsilon_\theta = \frac{u}{r}$; $\varepsilon_z = \frac{\partial w}{\partial z}$

$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$; are the ~~stress~~ strain displacement relationships

The Stress Strain relation is given by,

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

Displacement function are given by,



$$u(r,z) = a_1 + a_2 r + a_3 z$$

$$w(r,z) = a_4 + a_5 r + a_6 z$$

$$\therefore \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\& \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_{23} & a_{31} & a_{12} \\ b_{23} & b_{31} & b_{12} \\ \cancel{B} c_{32} & c_{13} & c_{21} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

&

$$\Rightarrow \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_{23} & a_{31} & a_{12} \\ b_{23} & b_{31} & b_{12} \\ c_{32} & c_{13} & c_{21} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}.$$

where,

$a_{12} = r_1 z_2 - r_2 z_1$; $b_{12} = z_1 - z_2$; $c_{21} = r_2 - r_1$

$a_{23} = r_2 z_3 - r_3 z_2$; $b_{23} = z_2 - z_3$; $c_{32} = r_3 - r_2$

$a_{31} = r_3 z_1 - r_1 z_3$; $b_{31} = z_3 - z_1$; $c_{13} = \cancel{\phi} r_1 - r_3$

and ~~a~~ shape functions ~~are~~ are,

$$N_1 = \frac{1}{2A}\left(a_{23} + b_{23} r + c_{32} z\right)$$

$$N_2 = \frac{1}{2A}\left(a_{31} + b_{31} r + c_{12} z\right)$$

$$N_3 = \frac{1}{2A}\left(a_{12} + b_{12} r + c_{21} z\right)$$

The Displacements can be rewritten as,

$$u_{\theta} = N_1 u_1 + N_2 u_2 + N_3 u_3$$

& $$w = N_1 w_1 + N_2 w_2 + N_3 w_3$$

∴ the strain matrix we get as,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_{23} & 0 & b_{31} & 0 & b_{12} & 0 \\ 0 & c_{32} & 0 & c_{13} & 0 & c_{21} \\ \frac{a_{23}}{\bar{r}} + b_{23} + \frac{c_{32}\bar{z}}{\bar{r}} & 0 & \frac{a_{31}}{\bar{r}} + b_{31} + \frac{c_{13}\bar{z}}{\bar{r}} & 0 & \frac{a_{12}}{\bar{r}} + b_{12} + \frac{c_{21}\bar{z}}{\bar{r}} & 0 \\ c_{32} & b_{23} & c_{13} & b_{31} & c_{21} & b_{12} \end{bmatrix} \begin{bmatrix} U_1 \\ N_1 \\ U_2 \\ W_2 \\ U_3 \\ W_3 \end{bmatrix}$$

The stress is given by

$$\{\sigma\} = [D][B][u]$$

and therefore the stiffness matrix is given by

$$[K] = \iiint_V [B]^T [D] [B] \, dV$$

$$= 2\pi \iint_A [B]^T [D] [B] \, r \, dr \, dz$$

We ~~start~~ evaluate $[B]$ for centroid point $(\bar{r}, \bar{z})$ of the element, getting,

$$r = \bar{r} = \frac{r_1 + r_2 + r_3}{3} \quad \& \quad z = \frac{z_1 + z_2 + z_3}{3}$$

and therefore $[k]$ is approximated as,

$$[k] = 2\pi \bar{r} A [\bar{B}]^T [D] [\bar{B}]$$

where

$$[\bar{B}] = [B(\bar{r}, \bar{z})]$$

Referenced From:
A First Course in the Finite Element Method
by Daryl L. Logan
Chapter - 9

Assignment 4.1

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1. $K^e = 2\pi\bar{r}\, A\, [\bar{B}^T][D][\bar{B}]$

where $\bar{r} = \dfrac{r_1 + r_2 + r_3}{3} = \dfrac{0 + a + a}{3} = \dfrac{2a}{3}$

$\bar{z} = \dfrac{z_1 + z_2 + z_3}{3} = \dfrac{0 + 0 + b}{3} = \dfrac{b}{3}$

$$[D] = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \because \nu = 0$$

$$[\bar{B}] = \frac{1}{2A}\begin{bmatrix} b_{23} & 0 & b_{31} & 0 & b_{12} & 0 \\ 0 & c_{32} & 0 & c_{13} & 0 & c_{21} \\ \dfrac{a_{23}}{\bar{r}} + b_{23} + \dfrac{c_{32}\bar{z}}{\bar{r}} & 0 & \dfrac{a_{31}}{\bar{r}} + b_{31} + \dfrac{c_{13}\bar{z}}{\bar{r}} & 0 & \dfrac{a_{12}}{\bar{r}} + b_{12} + \dfrac{c_{21}\bar{z}}{\bar{r}} & 0 \\ c_{32} & b_{23} & c_{13} & b_{31} & c_{21} & b_{12} \end{bmatrix}$$

$a_{12} = r_1 z_2 - r_2 z_1 = 0 - a.0 = 0$;

$a_{23} = r_2 z_3 - r_3 z_2 = a.b - a.0 = ab$;

$a_{31} = r_3 z_1 - r_1 z_3 = a.0 - 0.b = 0$;

$b_{12} = z_1 - z_2 = 0$; $\quad c_{21} = r_2 - r_1 = a$

$b_{23} = z_2 - z_3 = -b$; $\quad c_{32} = r_3 - r_2 = 0$

$b_{31} = z_3 - z_1 = b$; $\quad c_{13} = 0 - a = -a$

$$[B] = \frac{1}{2A}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{b}{2} & 0 & \frac{b}{2} & 0 & \frac{b}{2} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$\cancel{[D] = I}$

$$\therefore [k] = 2\pi \bar{r} A [B^T][D][B]$$

$$= \cancel{2\pi} \times \frac{\cancel{2}a}{3} \times \frac{\cancel{A}}{\cancel{4}A^2} \begin{bmatrix} -b & 0 & \frac{b}{2} & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & b/2 & -a \\ 0 & -a & 0 & b \\ 0 & 0 & b/2 & a \\ 0 & a & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\times \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$= \frac{2\pi}{3b} \begin{bmatrix} -b & 0 & b/2 & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & b/2 & -a \\ 0 & -a & 0 & b \\ 0 & 0 & b/2 & a \\ 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b/2 & -a/2 & b/2 & a/2 & 0 \end{bmatrix}$$

$$[K^e] = \frac{2\pi E}{3b}\begin{bmatrix} \frac{5b^2}{4} & 0 & -\frac{3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ 0 & \frac{b^2}{2} & \frac{ab}{2} & -\frac{b^2}{2} & -\frac{ab}{2} & 0 \\ -\frac{3b^2}{4} & \frac{ab}{2} & \frac{a^2}{2}+\frac{5b^2}{4} & -\frac{ab}{2} & \frac{b^2}{4}-\frac{a^2}{2} & 0 \\ 0 & -\frac{b^2}{2} & -\frac{ab}{2} & a^2+\frac{b^2}{2} & \frac{ab}{2} & -a^2 \\ \frac{b^2}{4} & -\frac{ab}{2} & -\frac{a^2}{2}+\frac{b^2}{4} & \frac{ab}{2} & \frac{b^2}{4}+\frac{a^2}{2} & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

2) Sum of 2nd Row $= \cancel{\frac{5b^2}{4}}\ \frac{b^2}{2} + \frac{ab}{2} - \frac{b^2}{2} - \frac{ab}{2} = 0$

" " 4th Row $= -\frac{b^2}{2} - \frac{ab}{2} + a^2 + \frac{b^2}{2} + \frac{ab}{2} - a^2 = 0$

" " 6th Row $= -a^2 + a^2 = 0$

Sum of 2nd column $= \frac{b^2}{2} + \frac{ab}{2} - \frac{b^2}{2} - \frac{ab}{2} = 0$

" " 4th column $= -\frac{b^2}{2} - \frac{ab}{2} + a^2 + \frac{b^2}{2} + \frac{ab}{2} - a^2 = 0$

" " 6th column $= \cancel{\frac{b^2}{4} - \frac{ab}{2} + \frac{b^2}{4} - \frac{a^2}{2} + \frac{ab}{2} + \frac{b^2}{4} + \frac{a^2}{2}} =$

$= -a^2 + a^2 = 0$

The sums are zero because if we take say unit displacement at each vertical node (w_i) the body will act like a rigid body and move along the vertical (z) axis. This will result in zero forces along the

~~vet~~ vertical direction. Hence the rows (and columns) are summed to be zero.

→ Sum of 1st row (and column) $= \frac{5b^2}{4} - \frac{3b^2}{4} + \frac{b^2}{4} = \frac{3b^2}{4}$

Sum of 3rd row (and column) $= -\frac{3b^2}{4} + \frac{ab}{2} + \frac{a^2}{2} + \frac{5b^2}{4}$

$$- \frac{ab}{2} - \frac{a^2}{2} + \frac{b^2}{4}$$

$$= \frac{3b^2}{4}$$

Sum of 5th row (and column) $= \frac{b^2}{4} - \frac{ab}{2} + \frac{b^2}{4} - \frac{a^2}{2} + \frac{ab}{2} + \frac{b^2}{4} + \frac{a^2}{2}$

$$= \frac{3b^2}{4}$$

The sum of rows (and columns) ~~do~~ of 1, 3 and 5 do not ~~vad~~ vanish because if we displace the horizontal components (u_i) by the same displacement say 1, then it is like expanding the surface of ~~so~~ revolution by 1 which ~~o~~ should inturn produce forces as shown by non-zero addition.

Considering force at a node say ~~b~~ i,

we have,

$$\{f_{bi}\} = 2\pi \iint_A [N_i]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\,dr\,dz \qquad \text{—(1)}$$

where $[N_i]^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$

where $R_b = \omega^2 \rho r$

and Z_b = body force per unit volume.

Considering centroid of triangle ~~at~~ and integrating,

we get,

$$\{f_{bi}\} = 2\pi \bar{r} A \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \bar{R}_b \\ Z_b \end{bmatrix}$$

$$= \frac{2\pi \bar{r} A}{3} \begin{bmatrix} \bar{R}_b \\ Z_b \end{bmatrix}$$

given, $\bar{R}_b = 0$ & $Z_b = -\rho g$

Hence we get

$$\{f^e\} = \{f_b\} = \frac{2\pi \bar{r} A}{3} \begin{Bmatrix} 0 \\ -\rho g \\ 0 \\ -\rho g \\ 0 \\ -\rho g \end{Bmatrix}$$