Computational Structural Mechanics and Dynamics

Assignment 4

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Assignment 4.1

1. Compute the entries of K^e for the following axisymmetric triangle:

$$r_1 = 0 \qquad r_2 = r_3 = a \qquad z_1 = z_2 = 0 \qquad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Solution

The shape functions for a triangular element in natural coordinates are written as:

$$N_1 = 1 - \xi - \eta = \xi_1 \ ,$$
$$N_2 = \xi = \xi_2 \ ,$$
$$N_3 = \eta = \xi_3.$$

The B matrix is found based on the following:

$$\boldsymbol{B} = \boldsymbol{DN}$$

where N is the matrix of shape functions and

$$D = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}$$

Therefore,

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

According to the definition of B, we have to differentiate the shape functions and r should be interpolated from the nodal coordinates. Therefore we have:

$$r = \sum r_i N_i = a(\xi + \eta) = a(\xi_2 + \xi_3)$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_2}{\partial \xi} \\ \frac{\partial N_2}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} \\ -\frac{1}{b} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{b} \end{bmatrix}$$

Therefore,

$$\boldsymbol{B} = \begin{bmatrix} \frac{-1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} \\ \frac{1-\xi-\eta}{a(\xi+\eta)} & \frac{\xi}{a(\xi+\eta)} & \frac{\eta}{a(\xi+\eta)} & 0 & 0 & 0 \\ 0 & \frac{-1}{b} & \frac{1}{b} & \frac{-1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

And in terms of ξ_1, ξ_2 and ξ_3 we have:

$$\boldsymbol{B} = \begin{bmatrix} \frac{-1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} \\ \frac{\xi_1}{a(\xi_2+\xi_3)} & \frac{\xi_2}{a(\xi_2+\xi_3)} & \frac{\xi_3}{a(\xi_2+\xi_3)} & 0 & 0 & 0 \\ 0 & \frac{-1}{b} & \frac{1}{b} & \frac{-1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

According to the formula for calculating the stiffness matrix, for a quadrilateral element with p integration points we have:

$$K^{(e)} = \sum\sum 2\pi\omega_k\omega_l B^T(\xi_k\eta_l)EB(\xi_k\eta_l)r(\xi_k\eta_l)J(\xi_k\eta_l)$$

where J is the determinant of the jaccobian found as:

$$J = \sum \begin{bmatrix} \frac{\partial N_i}{\partial \xi} r_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} r_i & \frac{\partial N_i}{\partial \eta} z_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & b \end{bmatrix} \Longrightarrow det(J) = ab$$

Using Gauss centroid rule to compute the integration, we have to substitute $\xi_1 = \xi_2 = \xi_3 = \frac{1}{3}$ and $\omega_k = \omega_l = 0.5$ into the element stiffness matrix. Therefore we will obtain K as the following:

$$\boldsymbol{K} = \frac{2\pi abE}{2} \begin{bmatrix} \frac{5}{6a} & -\frac{1}{2a} & \frac{1}{6a} & 0 & 0 & 0 \\ -\frac{1}{2a} & \frac{2a}{3}((\frac{5}{4a^2}) + \frac{1}{2b^2}) & \frac{2a}{3}((\frac{1}{4a^2}) - \frac{1}{2b^2}) & \frac{1}{3b} & -\frac{1}{3b} & 0 \\ \frac{1}{6a} & \frac{2a}{3}((\frac{1}{4a^2}) - \frac{1}{2b^2}) & \frac{2a}{3}((\frac{1}{4a^2}) + \frac{1}{2b^2}) & -\frac{1}{3b} & \frac{1}{3b} & 0 \\ 0 & \frac{1}{3b} & -\frac{1}{3b} & \frac{1}{3a} & -\frac{1}{3a} & 0 \\ 0 & -\frac{1}{3b} & \frac{1}{3b} & -\frac{1}{3a} & \frac{2a}{3}((\frac{1}{2a^2}) + \frac{1}{b^2}) & -\frac{2a}{3b^2} \\ 0 & 0 & 0 & 0 & -\frac{2a}{3b^2} & \frac{2a}{3b^2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $K^{(e)}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.
   Solution
   As it can be seen in K matrix obtained in the previous question, the sum of the elements of rows (columns) 2 and 4 and 6 which are related to degree of freedom in z-direction is zero. This is because in this direction we have rigid body motion. However, the sum of the elements related to r-direction is not zero since it is not experiencing rigid-body motion and is restricted in this direction.

3. Compute the consistent force vector $f^{(e)}$ for gravity forces $b = [0, -g]^T$.
Solution
to compute the force vector, we have:

$$f_{ext}^{(e)} = \sum \sum 2\pi \omega_k \omega_l N^T(\xi_k \eta_l) b(\xi_k \eta_l) r(\xi_k \eta_l) J(\xi_k \eta_l)$$

where J is the determinant of the jaccobian (which has been computed previously) and N is found as below:

$$N = \begin{bmatrix} N_1^{(e)} & N_2^{(e)} & N_3^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^{(e)} & N_2^{(e)} & N_3^{(e)} \end{bmatrix} \Longrightarrow N = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \end{bmatrix}$$

Using Gauss centroid rule, we have:

$$f = \frac{2\pi abE}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{2ag}{9} \\ -\frac{2ag}{9} \\ -\frac{2ag}{9} \end{bmatrix}$$