On "Convergence requirements"

Assignment 5.1

The isoparametric definition of the straight–node bar element in its local system $\underline{x}$ is,

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values −1, 1 and 0 at nodes 1, 2 and 3 respectively, while N^e_1, N^e_2 and N^e_3 are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0, \bar{x}_2 = L, \bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum α (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are ±¼ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

Date of Assignment: **9 / 03 / 2020**
Date of Submission: **16 / 03 / 2020**

The assignment must be submitted as a pdf file named **As5-Surname.pdf** to the CIMNE virtual center.

(1) Assignment 5 Andino Saint Antonin

(5.1) The shape functions for the three-node bar element are (from Assignment 4)

$$\bar{N} = \left[\frac{\varepsilon}{2}(\varepsilon - 1) \quad \frac{\varepsilon}{2}(\varepsilon + 1) \quad 1 - \varepsilon^2 \right]$$

we also obtained

$$x = \sum_{i=1}^{3} x_i N_i = \frac{\varepsilon l}{2}(1+\varepsilon) + l\left(\frac{l}{2} + \alpha\right)(1-\varepsilon^2)$$

and the Jacobian $\left(\frac{dx}{d\varepsilon}\right)$

$$J = \frac{l}{2}(1 - 4\alpha\varepsilon)$$

the Jacobian takes value zero for

$$J = \frac{dx}{d\varepsilon} = \frac{l}{2}(1 - 4\alpha\varepsilon) = 0$$

$$1 - 4\alpha\varepsilon = 0$$

$$\alpha\varepsilon = -\frac{1}{4}$$

$$\varepsilon = 1 \Rightarrow \alpha = -1/4$$

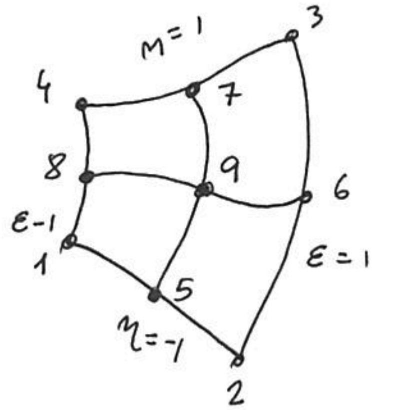
$$\varepsilon = -1 \Rightarrow \alpha = 1/4$$

since $e = B u^e = \frac{dN}{dx} u^e = J^{-1} \frac{dN}{d\varepsilon} u^e$

but J^{-1} does not exist! the strain becomes infinite for $|\alpha| = 1/4$

(2)

5.2 9-node plane stress element:



The shape function vector is

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1-\varepsilon)(1-\eta)\varepsilon\eta \\ -\frac{1}{4}(1+\varepsilon)(1-\eta)\varepsilon\eta \\ \frac{1}{4}(1+\varepsilon)(1+\eta)\varepsilon\eta \\ -\frac{1}{4}(1-\varepsilon)(1+\eta)\varepsilon\eta \\ -\frac{1}{2}(1-\varepsilon^2)(1-\eta)\eta \\ \frac{1}{2}(1+\varepsilon)(1-\eta^2)\varepsilon \\ \frac{1}{2}(1-\varepsilon^2)(1+\eta)\eta \\ -\frac{1}{2}(1-\varepsilon)(1+\eta^2)\varepsilon \\ (1-\varepsilon^2)(1-\eta^2) \end{bmatrix}$$

where the shape functions vanish in all nodes except their source node, where their value is 1

In order to differentiate w.r.t. the isoparametric variables, we must apply the chain rule

$$\begin{bmatrix} \frac{\partial}{\partial \varepsilon} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial y}{\partial \varepsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \bar{\bar{J}} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

which also means:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \bar{\bar{J}}^{-1} \begin{bmatrix} \frac{\partial}{\partial \varepsilon} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

(3) considering two coordinates (x and y)

$$x(\varepsilon, \eta) = \sum_{i=1}^{9} x_i N_i(\varepsilon, \eta)$$

$$y(\varepsilon, \eta) = \sum_{i=1}^{9} y_i N_i(\varepsilon, \eta)$$

then the Jacobian becomes

$$J = \begin{bmatrix} \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \varepsilon} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \varepsilon} \\ \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Next, we must compute 18 derivatives $\left(\frac{\partial N_i}{\partial \varepsilon} \text{ and } \frac{\partial N_i}{\partial \eta}\right.$ and evaluate them at the x_i, y_i point corresponding to each node.

Using Wolfram Alpha for this Tedious task we get:

$$J = \begin{bmatrix} L\left(\frac{1}{2} - 2\alpha\right) & 0 \\ 0 & \frac{L}{2} \end{bmatrix}$$

where $-1/2 < \alpha < 1/2$ and $\alpha = 0$ for a perfect square. We will have a singularity (infinit. strain) is J^{-1} does not exist.

or equivalently, if $\det(J) = 0$

(4)

$$\det(J) = \frac{L^2}{2}\left(\frac{1}{2} - 2\alpha\right) = 0$$

$$\boxed{\alpha = \frac{1}{4}}$$

since $\quad \bar{e} = \bar{\bar{B}}\,\bar{u} = \overline{\frac{dN_i}{dx}}\,\bar{u} = J^{-1}\,\overline{\frac{dN_i}{d\xi}}\,\bar{u}$

for $\alpha = 1/4$ there is no J^{-1} and the strain is undefined (singularity)