Assignment 5

→ Assignment 5.1 [Isoparametric Representation]

Q) Three node bar element.

Shape functions - $N_1(\xi)$, $N_2(\xi)$ & $N_3(\xi)$

ie,
$$N_1^e(\xi) = a_0 + a_1\xi_e + a_2\xi_e^2$$
$$N_2^e(\xi) = b_0 + b_1\xi_e + b_2\xi_e^2$$
$$N_3^e(\xi) = c_0 + c_1\xi_e + c_2\xi_e^2$$

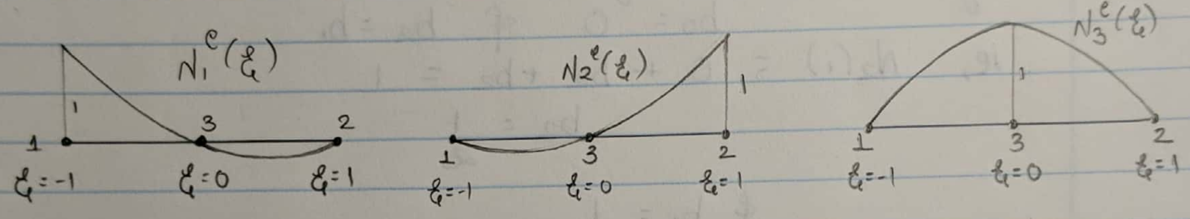


Fig. Isoparametric shape functions for 3 node bar element

Q.a) Coefficients $a_0 \ldots c_2$, values?

→ Solution:

(i) Node 1: we know that $N_1^e = 1$ for $\xi_e = -1$ & for rest ~~is~~ its 0.

$$N_1(\xi_e) = a_0 + a_1\xi_e + a_2\xi_e^2$$

substitute $\xi_e = -1, 0$ & 1.

$$N_1(-1) = a_0 - a_1 + a_2 = 1$$
$$N_1(0) = a_0 = 0$$
$$N_1(1) = a_0 + a_1 + a_2 = 0$$

from above eq^n :-

$$a_0 = 0, \quad a_2 = -a_1$$

ie, $N_1(-1) = 0 - a_1 - a_1 = 1$

$$a_1 = \frac{-1}{2}$$

$$\& \quad a_2 = \frac{1}{2}$$

ie, $\boxed{a_0 = 0, \quad a_1 = \frac{-1}{2} \quad \& \quad a_2 = \frac{1}{2}}$

(ii) Node 2 :

$N_2 = 1$ for $\xi_l = 1$ and 0 for rest.

$N_2(\xi_l) = b_0 + b_1 \xi_l + b_2 \xi_l^2$.

substitute. $\xi_l = -1, 0 \ \& \ 1$.

$N_2(-1) = b_0 - b_1 + b_2 = 0$

$N_2(0) = b_0 = 0$

$N_2(1) = b_0 + b_1 + b_2 = 1$

from above equations,

$b_0 = 0 \ \& \ b_2 = b_1$

ie, $N_2(1) = 0 + b_2 + b_2 = 1$

$b_2 = \frac{1}{2}$

$\& \ b_1 = \frac{1}{2}$

ie, $\boxed{b_0 = 0, \ b_1 = \frac{1}{2} \ \& \ b_2 = \frac{1}{2}}$

(iii) Node 3 :

$N_3 = 1$ for $\xi_l = 0$ and 0 for rest.

$N_3(\xi_l) = C_0 + C_1 \xi_l + C_2 \xi_l^2$.

substitute $\xi_l = -1, 0 \ \& \ 1$

$N_3(-1) = C_0 - C_1 + C_2 = 0$

$N_3(0) = C_0 = 1$

$N_3(1) = C_0 + C_1 + C_2 = 0$

from above equations,

$C_0 = 1, \ 1 + C_1 = -C_2 \Rightarrow C_2 = C_1 - 1$

ie, $N_3(1) = 1 + C_1 + C_1 - 1 = 0$

$2C_1 = 0$

$C_1 = 0$

$\& \ C_2 = 0 - 1 = -1$

ie, $\boxed{C_0 = 1, \ C_1 = 0 \ \& \ C_2 = -1}$

Q.b) Verify that their sum is identically 1.
$(N_1 + N_2 + N_3 = 1)$

→ Solution:
To add all the shape functions.

$$N_1(\xi) + N_2(\xi) + N_3(\xi) = (a_0 + a_1\xi + a_2\xi^2) + (b_0 + b_1\xi + b_2\xi^2) + (c_0 + c_1\xi + c_2\xi^2)$$

substitute the coefficient values,

$$N_1 + N_2 + N_3 = \left(0 - \frac{1}{2}\xi + \frac{1}{2}\xi^2\right) + \left(0 + \frac{1}{2}\xi + \frac{1}{2}\xi^2\right) + (1 + 0 - \xi^2)$$

$$= \cancel{-\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + \cancel{\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + 1 - \cancel{\xi^2}$$

$$N_1 + N_2 + N_3 = 1$$

ie. $\boxed{N_1(\xi) + N_2(\xi) + N_3(\xi) = 1}$

Qc) Calculate the derivatives with respect to natural co-ordinates. (ξ)

→ Solution:

(i) $N_1 = \frac{-1}{2}\xi + \frac{1}{2}\xi^2$

$$\frac{\partial N_1}{\partial \xi} = \frac{-1}{2} + \xi$$

ie. $\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2}$

(ii) $N_2 = \frac{1}{2}\xi + \frac{1}{2}\xi^2$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{2} + \xi$$

ie, $\frac{\partial N_2}{\partial \xi} = \xi + \frac{1}{2}$

(iii) $N_3 = 1 - \xi_1^2$

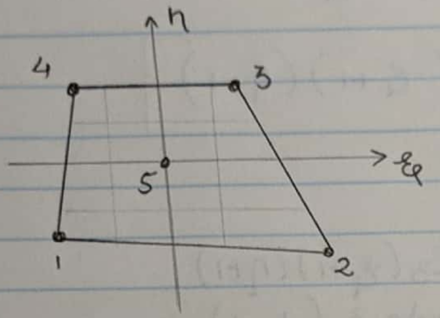
$$\frac{\partial N_3}{\partial \xi_1} = -2\xi_1$$

ie, $$\frac{\partial N_3}{\partial \xi_1} = -2\xi_1.$$

Ans: $\boxed{\frac{\partial N_1}{\partial \xi_1} = \xi_1 - \frac{1}{2}, \quad \frac{\partial N_2}{\partial \xi_1} = \xi_1 + \frac{1}{2} \quad \& \quad \frac{\partial N_3}{\partial \xi_1} = -2\xi_1}$

→ Assignment 5.2 [Isoparametric Representation]

Q.) Find shape function (N_i^e) that satisfy compatibility and to verify that their sum is unity.



→ Solution.

ⓐ To find the shape functions (N_i^e), ($i = 1, 2, 3, 4, 5$)
Node 1, 2, 3 & 4 are corner node while node 5 is the centered node.

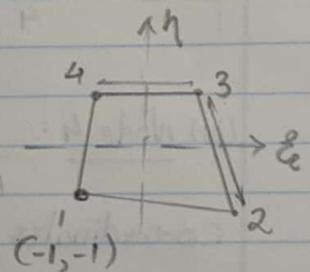
Corner Node -

(i) Node 1 :

$\bar{N}_1 = C_1 L_{23} \cdot L_{34} = C_1(\xi-1)(\eta-1)$

coordinates of node 1 (−1, −1)

ie, $\xi = -1$, $\eta = -1$



$\bar{N}_1 = C_1(-1-1)(-1-1) = 1$

$C_1(-2)(-2) = 1$

$C_1 = \frac{1}{4}$

$$\boxed{\bar{N}_1 = \frac{1}{4}(\xi-1)(\eta-1)}$$

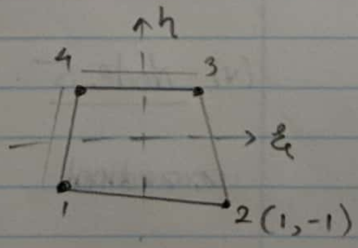
(ii) Node 2 :

$\bar{N}_2 = C L_{34} L_{41} = C_2(\xi+1)(\eta-1)$

~~node 2 (ξ, η) is~~

Co-ordinates of node 2 (1, −1)

$\xi = 1$ & $\eta = -1$



$$\bar{N}_2 = C_2(1+1)(-1-1) = 1$$
$$C_2(2)(-2) = 1$$
$$C_2 = -\frac{1}{4}$$

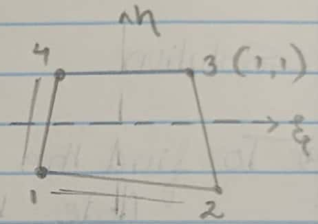
$$\boxed{\bar{N}_2 = -\frac{1}{4}(\xi+1)(\eta-1)}$$

(iii) Node 3 :

$$\bar{N}_3 = C_3(\xi+1)(\eta+1)$$

coordinates of node 3 (1, 1)

$\xi = 1$ & $\eta = 1$



$$\bar{N}_3 = C_3(1+1)(1+1) = 1$$
$$C_3(2)(2) = 1$$
$$C_3 = \frac{1}{4}$$

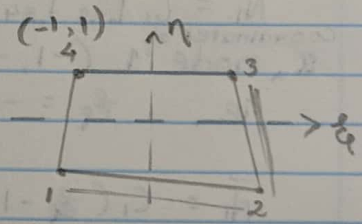
$$\boxed{\bar{N}_3 = \frac{1}{4}(\xi+1)(\eta+1)}$$

(iv) Node 4 :

$$\bar{N}_4 = C_4(\xi-1)(\eta+1)$$

Co-ordinates of node 4 (-1, 1)

$\xi = -1$ & $\eta = 1$



$$\bar{N}_4 = C_4(-1-1)(1+1) = 1$$
$$C_4(-2)(2) = 1$$
$$C_4 = \frac{-1}{4}$$

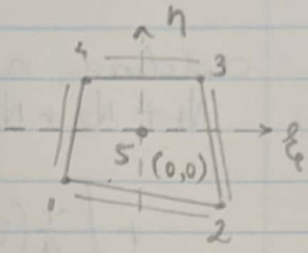
$$\boxed{\bar{N}_4 = \frac{-1}{4}(\xi-1)(\eta+1)}$$

(v) Node 5 :

$$N_5 = C_5(\xi-1)(\eta-1)(\xi+1)(\eta+1)$$

~~coordinat~~

$$N_5 = C_5(\xi^2-1)(\eta^2-1)$$



coordinate of node 5 (0.0)

$\xi_e = 0, \eta = 0$

$N_5 = C_5(0-1)(0-1) = 1$

$C_5(-1)(-1) = 1$

$C_5 = 1$

$$\boxed{N_5 = (\xi_e^2 - 1)(\eta^2 - 1)}$$

(b) Determining α?

formula: $N_i = \bar{N}_i + \alpha N_5$.

Using coordinate of node 5 (0.0)

$$0 = \frac{1}{4}(\xi_e - 1)(\eta - 1) + \alpha(\xi_e^2 - 1)(\eta^2 - 1)$$

$$0 = \frac{1}{4}(-1)(-1) + \alpha(-1)(-1)$$

$$\boxed{\alpha = -\frac{1}{4}}$$

(c) To verify that their sum is 1 (unity).

shape functions are redefined, using formula $N_i = \bar{N}_i + \alpha N_5$.

$$N_1 = \bar{N}_1 + \alpha N_5 = \frac{1}{4}(\xi_e - 1)(\eta - 1) - \frac{1}{4}(\xi_e^2 - 1)(\eta^2 - 1)$$

$$N_2 = \bar{N}_2 + \alpha N_5 = -\frac{1}{4}(\xi_e + 1)(\eta - 1) - \frac{1}{4}(\xi_e^2 - 1)(\eta^2 - 1)$$

$$N_3 = \bar{N}_3 + \alpha N_5 = \frac{1}{4}(\xi_e + 1)(\eta + 1) - \frac{1}{4}(\xi_e^2 - 1)(\eta^2 - 1)$$

$$N_4 = \bar{N}_4 + \alpha N_5 = \frac{1}{4}(\xi_e - 1)(\eta + 1) - \frac{1}{4}(\xi_e^2 - 1)(\eta^2 - 1)$$

$$N_5 = (\xi_e^2 - 1)(\eta^2 - 1)$$

adding all shape functions –

$$N_1 + N_2 + N_3 + N_4 + N_5 = \left[\frac{1}{4}(\xi-1)(\eta-1) - \frac{1}{4}(\xi^2-1)(\eta^2-1)\right] +$$

$$+ \left[\frac{-1}{4}(\xi+1)(\eta-1) - \frac{1}{4}(\xi^2-1)(\eta^2-1)\right] + \left[\frac{1}{4}(\xi+1)(\eta+1) - \frac{1}{4}(\xi^2-1)(\eta^2-1)\right] +$$

$$+ \left[\frac{-1}{4}(\xi-1)(\eta+1) - \frac{1}{4}(\xi^2-1)(\eta^2-1)\right] + \left[(\xi^2-1)(\eta^2-1)\right]$$

on simplifying,

$$N_1+N_2+N_3+N_4+N_5 = \cancel{\frac{\xi\eta}{4}} - \cancel{\frac{\xi}{4}} - \cancel{\frac{\eta}{4}} - \frac{\xi^2\eta^2}{4} + \frac{\xi^2}{4} + \frac{\eta^2}{4} - \cancel{\frac{\xi\eta}{4}} + \cancel{\frac{\xi}{4}} - \cancel{\frac{\eta}{4}} - \frac{\xi^2\eta^2}{4} +$$

$$+ \frac{\xi^2}{4} + \frac{\eta^2}{4} + \cancel{\frac{\xi\eta}{4}} + \cancel{\frac{\xi}{4}} + \cancel{\frac{\eta}{4}} - \frac{\xi^2\eta^2}{4} + \frac{\xi^2}{4} + \frac{\eta^2}{4} - \cancel{\frac{\xi\eta}{4}} - \cancel{\frac{\xi}{4}} + \cancel{\frac{\eta}{4}} - \frac{\xi^2\eta^2}{4} +$$

$$+ \frac{\xi^2}{4} + \frac{\eta^2}{4} + \xi^2\eta^2 - \xi^2 - \eta^2 + 1$$

$$N_1 + N_2 + N_3 + N_4 + N_5 = -\cancel{4}\cancel{\frac{\xi^2\eta^2}{4}} + \cancel{\frac{4\xi^2}{4}} + \cancel{\frac{4\eta^2}{4}} + \cancel{\xi^2\eta^2} - \cancel{\xi^2} - \cancel{\eta^2} + 1$$

ie. $$\boxed{N_1 + N_2 + N_3 + N_4 + N_5 = 1}$$

→ Assignment 5.3 [Convergence Requirements]

Q) which minimum integration rules of gauss product type gives a rank sufficient stiffness matrix for the given element.
8 node hexahedron, 20 node hexahedron, 27 node hexahedron & 64 node hexahedron.

→ Solution:

(a) ~~8 node hexahedron~~ :-

Rank Sufficient: $r = n_F - n_R$

For numerically integrating,

$$r = \min(n_F - n_R, n_E n_g)$$

ie, for rank sufficiency,

$$n_E n_g \geq n_F - n_R.$$

Using this equation we can select gauss integration product rule.

where,
- n - node
- n_F - number of element degrees of freedom ($3n$)
- n_R - number of independent rigid body modes.
- r - rank of stiffness matrix.
- n_g - number of gauss points.
- n_E - order of stress strain matrix E

Here, Hexahedron element is considered.
Each element has 3 degrees of freedom & Independent rigid body modes are 6. Stress strain matrix E is full rank, plane stress is considered. So E is 3×3 matrix.

(a) 8 node Hexahedron -

$n = 8$, $n_F = 24$, $n_R = 6$, $n_E = 3$.

rank sufficiency $(r) = n_F - n_R = 24 - \cancel{8}6 = 18$.

Rank sufficiency condition : $n_E n_g \geq n_F - n_R$

ie, $3n_g \geq 18$

$n_g \geq 6$

$\therefore$, Required gauss rule to attain rank sufficiency is $2 \times 2 \times 2$ rule

ie. $\boxed{n_g = 8}$

(b) 20 node hexahedron -

$n = 20$, $n_F = 60$, $n_R = 6$, $n_E = 3$

$r = n_F - n_R = 60 - 6 = 54$

$\therefore$, Condition is -

$n_E n_g \geq n_F - n_R$

$3n_g \geq 54$

$n_g \geq 18$

Required gauss rule to attain rank sufficiency is, $3 \times 3 \times 3$ rule.

ie $\boxed{n_g = 27}$

(c) 27 node hexahedron -

$n = 27$, $n_F = 81$, $n_R = 6$, $n_E = 3$

$r = n_F - n_R = 81 - 6 = 75$

Condition is,

$n_E n_g \geq n_F - n_R$

$3n_g \geq 75$

$n_g \geq 25$

Required gauss rule to attain rank sufficiency is $3\times3\times3$ product rule.

ie. $\boxed{n_g = 27}$

ⓓ 64 node hexahedron –

$n = 64$, $\quad n_F = 192 \quad n_R = 6 \quad n_E = 3$

$r = n_F - n_R = 192 - 6 = 186$.

Condition for rank sufficiency is

$$n_E n_g \geq n_F - n_R$$
$$3n_g \geq 186$$
$$n_g \geq 62$$

Required gauss rule to attain rank sufficiency is $4\times4\times4$ product rule.

ie. $\boxed{n_g = 64}$

Rank Sufficient Gauss Rule :

Element	n	n_F $(3n)$	$n_F - n_R$ $(n_R=6)$	Min n_g	Product Rule
8 node hexahedron	8	24	18	6	$2\times2\times2$
20 node hexahedron	20	60	54	18	$3\times3\times3$
27 node hexahedron	27	81	75	25	$3\times3\times3$
64 node hexahedron	64	192	186	62	$4\times4\times4$

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