UNIVERSITAT POLITÈCNICA DE CATALUNYA

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Titulació

Assignatura

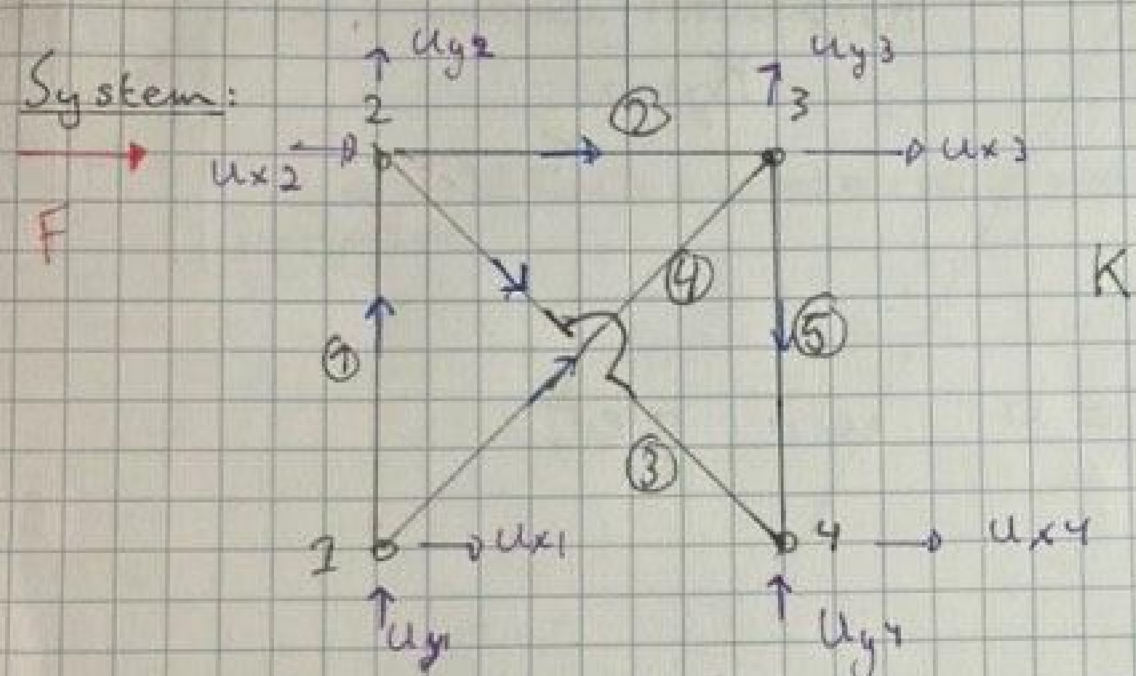
Cognoms | Nom | Pàgina ____ de ____

DNI

$L^{(3)} = L^{(4)} = 6\sqrt{2}\cdot 10^3 \text{ mm}$ $E = 200\,000 \text{ MPa}$, $F = 80000 \text{ N}$ $A = 600 \text{ mm}^2$

$L^{(1)} = L^{(2)} = L^{(5)} = 6000 \text{ mm}$

System:



In general:

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & cs & -c^2 & -sc \\ cs & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$c = \cos(\phi) \quad s = \sin(\phi)$

$$K^1 = 20000 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$\phi = \pi/2$

$$K^2 = 20000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\phi = 0$

$$K^3 = 7071 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$\phi = 3\pi/4$

$$K^4 = 7071 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$\pi/4$

Calculating everything in mm.

$$K^5 = 20000 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$\phi = \frac{3\pi}{2}$

Using equilibrium to assemble:

Element 1:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 20000 \begin{bmatrix} \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Element 2:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 20000 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & & & 0 & & & 0 \\ 0 & 0 & & & & & & \end{bmatrix} \vec{u}$$

Element 3:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 7071 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \cdot \vec{u}$$

Element 4:

$$\vec{f} = 7071 \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & \\ 1 & 1 & 0 & 0 & -1 & -1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ -1 & -1 & 0 & 0 & 1 & 1 & \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \vec{u}$$

Element 5:

$$\vec{f} = 20000 \begin{bmatrix} 0 & 0 \\ 0 & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{matrix} \end{bmatrix} \vec{u}$$

Adding all together:

$$\begin{bmatrix} 0 \\ 0 \\ F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7071 & 7071 & 0 & 0 & -7071 & -7071 & 0 & 0 \\ 7071 & 27071 & 0 & -20000 & -7071 & -7071 & 0 & 0 \\ 0 & 0 & 27071 & -7071 & -20000 & 0 & -7071 & 7071 \\ 0 & -20000 & -7071 & 27071 & 0 & 0 & 7071 & -7071 \\ -7071 & -7071 & -20000 & 0 & 27071 & 7071 & 0 & 0 \\ -7071 & -7071 & 0 & 0 & 7071 & 27071 & 0 & -20000 \\ 0 & 0 & -7071 & 7071 & 0 & 0 & 7071 & -7071 \\ 0 & 0 & 7071 & -7071 & -20000 & 0 & -7071 & 20000 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Applying boundary conditions:

$u_1 = v_1 = u_4 = v_4 = 0$ we get following

$$\begin{bmatrix} 80000 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 27071 & -7071 & -20000 & 0 \\ -7071 & 27071 & 0 & 0 \\ -20000 & 0 & 27071 & 7071 \\ 0 & 0 & 7071 & 27071 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

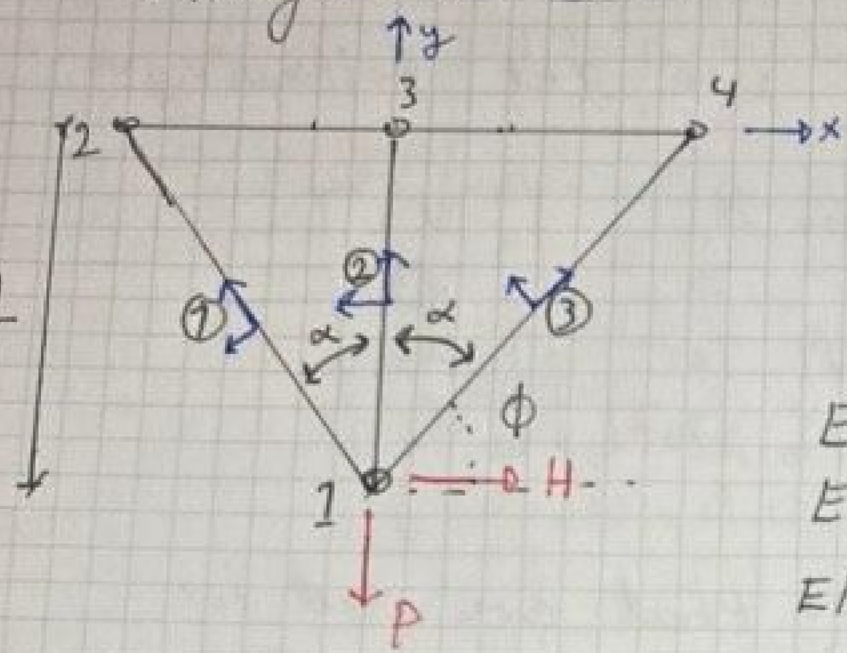
Deleting rows and colums that are used for u_1, v_1, u_4, v_4.

Which gives: Inverting 4x4 matrix on computer

$$\begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = 10^{-4} \begin{bmatrix} 1{,}068 & 0{,}279 & 0{,}847 & -0{,}221 \\ 0{,}279 & 0{,}442 & 0{,}221 & 0{,}058 \\ 0{,}847 & 0{,}221 & 1{,}068 & -0{,}279 \\ -0{,}221 & 0{,}058 & -0{,}279 & 0{,}442 \end{bmatrix} \begin{bmatrix} 80000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8{,}544 \\ 2{,}23 \\ 6{,}776 \\ -1{,}768 \end{bmatrix} \text{mm}$$

Assignment 1



- E, A same for all bars

$$L^{(2)} = L$$
$$L^{(1)} = L^{(3)} = \frac{L}{\cos(\alpha)} = \frac{L}{c}$$

Element 1 : $\phi = \pi/2 + \alpha$

Element 2 : $\phi = \pi/2$

Element 3 : $\phi = \pi/2 - \alpha$

a) Finding globalized element stiffnesses:

Using identities:

$$\sin(\pi/2 - x) = \cos(x)$$
$$\cos(\pi/2 - x) = \sin(x)$$

and
$$K^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & cs & -c^2 & -sc \\ cs & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$c = \cos(\phi) \quad s = \sin(\phi)$

Element 1 :

$$\phi = \pi/2 + \alpha \Rightarrow \begin{aligned} \cos(\phi) &= -\sin(\alpha) \\ \sin(\phi) &= \cos(\alpha) \end{aligned}$$

$$K^{(1)} = \frac{EA}{L} \cdot c \begin{bmatrix} s^2 & -cs & -s^2 & sc \\ & c^2 & sc & -c^2 \\ & & s^2 & -sc \\ \text{Symm} & & & c^2 \end{bmatrix}$$

$$= \frac{EA}{L}\begin{bmatrix} cs^2 & -c^2 s & -s^2 c & sc^2 \\ & c^3 & sc^2 & -c^3 \\ & & s^2 c & -sc^2 \\ & & & c^3 \end{bmatrix}$$

Element 2:

$$K^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 3:

$$\phi = \pi/2 - \alpha \Rightarrow \cos(\phi) = \sin(\alpha)$$
$$\sin(\phi) = \cos(\alpha)$$

$$K^{(3)} = \frac{EA}{L} \begin{bmatrix} cs^2 & c^2s & -s^2c & -sc^2 \\ & c^3 & -sc^2 & -c^3 \\ & & s^2c & sc^2 \\ \text{Sym.} & & & c^3 \end{bmatrix}$$

Expand to master stiffness:

Element 1:

$$\frac{EA}{L} \left[\begin{array}{cccc|c} cs^2 & -c^2s & -s^2c & sc^2 & \\ -c^2s & c^3 & sc^2 & -c^3 & 0 \\ -s^2c & sc^2 & s^2c & -sc^2 & \\ sc^2 & -c^3 & -c^3 & c^3 & \\ \hline & & 0 & & 0 \end{array}\right]$$

Element 2:

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3:

$$\frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & & & -s^2c & -sc^2 \\ c^2s & c^3 & & & -sc^2 & -c^3 \\ & 0 & & 0 & & 0 \\ -s^2c & -sc^2 & & & s^2c & sc^2 \\ -sc^2 & -c^3 & & & sc^2 & c^3 \end{bmatrix}$$

Combined we get master stiffness matrix and eq:

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 2cs & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & \text{Sym} & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Since element 2 is oriented vertically, it does not have any stiffness in the global x-direction.
5th row and column are bound to u_{x3} and since there is no stiffness in this direction they are zero.

b) The boundary conditions are:

$$u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$$

Which gives:

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix}$$

c) Inverting we get:

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \frac{L}{EA} \cdot \frac{1}{2cs^2(1+2c^3)} \begin{bmatrix} 1+2c^3 & 0 \\ 0 & 2cs^2 \end{bmatrix} \begin{bmatrix} H \\ -P \end{bmatrix}$$

$$= \frac{L}{EA\, 2cs^2(1+2c^3)} \begin{bmatrix} H(1+2c^3) \\ -2Pcs^2 \end{bmatrix}$$

$$= \frac{L}{EA} \begin{bmatrix} H/2cs^2 \\ -P/(1+2c^3) \end{bmatrix}$$

$\alpha \to 0$:

$u_{x1} \to$ "$\dfrac{LH}{2EA \cdot 0}$" $\to$ "∞" which makes sense because if $\alpha = 0$ there is no resistance against horizontal movement and the bars will rotate freely about node 3.
$\Rightarrow$ u_{x1} blows up.

$$u_{y1} \to \frac{-LP}{3EA}$$

Hookes law: $\sigma = E\varepsilon = E \dfrac{\Delta L}{L}$

3 bars $\Rightarrow$ $\dfrac{P}{3A} = E \dfrac{\Delta L}{L} \Rightarrow \Delta L = \dfrac{PL}{3EA}$

$\Rightarrow$ The minus comes from elongation in the same direction as P $\Rightarrow$ Makes sense.

$\alpha \to \pi/2$:

$u_{x1} \to$ "$\dfrac{LH}{0}$" $\to$ "∞". $L^{(1)} = L^{(3)} \to$ '∞'

Element 1 and 3 cannot contribute to any stiffness in x-direction or y-direction
$\Rightarrow$ Free to rotate $\Rightarrow$ Makes sense.

$u_{y1} \to \dfrac{-PL}{EA}$ which makes sense because it is only one bar taking the force P this time.

d) Using the displacement transformation:

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

$c = \cos\phi \quad , s = \sin\phi$

Element 1: $\phi = \pi/2 + \alpha \Rightarrow \cos(\phi) = -\sin(\alpha)$
$\sin(\phi) = \cos(\alpha)$

$\bar{u}_{x1} = -s\,u_{x1} + c \cdot u_{y1}$ } $c = \cos(\alpha)$, $s = \sin(\alpha)$

This entongment will induce forces.

$d = \bar{U}_{x2} - \bar{U}_{x1}$ which gives

$$\frac{F}{A} = E \cdot \frac{\bar{u}_{x1}}{L} \cdot c$$

$$\Rightarrow F = \frac{AE}{L} \cdot c \cdot \left[+s \cdot \frac{H}{2cs^2} + c \cdot \frac{+P}{(1+2c^3)} \right] \cdot \frac{L}{EA}$$

$$= \left[\frac{+H}{2s} + \frac{+c^2 P}{(1+2c^3)} \right] \quad \text{(tension)}$$

Element 2: $\phi = \pi/2 \Rightarrow c = 0 \quad , s = 1$

$\Rightarrow \bar{u}_{x1} = u_{y1} \quad , d = -\bar{U}_{x1}$

$$\Rightarrow F = \frac{+P}{(1+2c^3)} \quad \text{(Tension)}$$

Element 3: $\phi = \pi/2 - \alpha \Rightarrow \cos\phi = \sin\alpha$

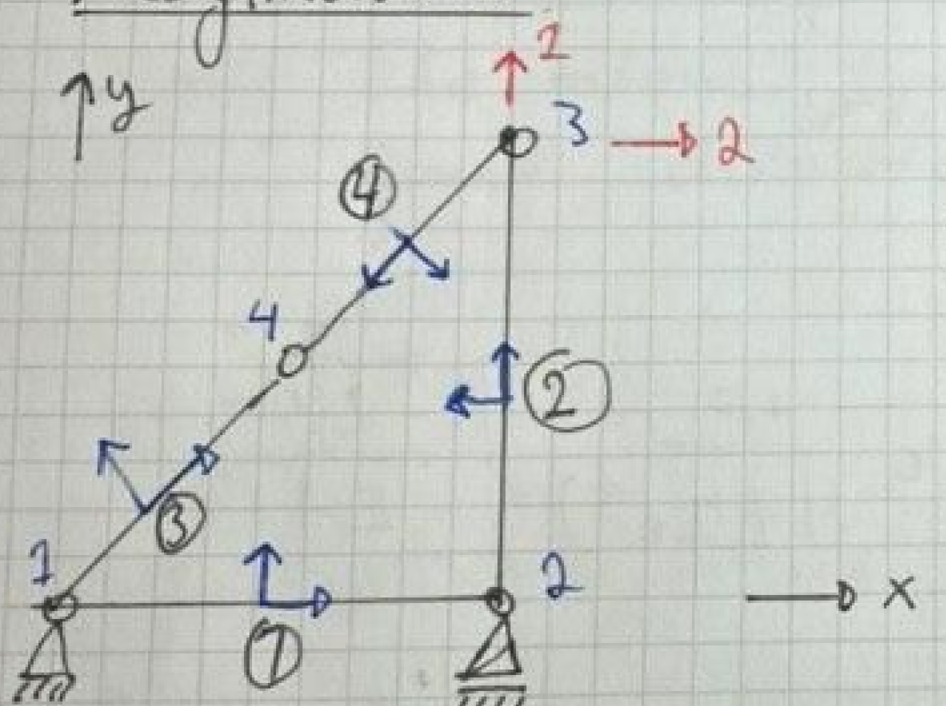
$\sin\phi = \cos\alpha$

$\Rightarrow \bar{u}_{x1} = s\, u_{x1} + c \cdot u_{y1}$ } $c = \cos\alpha$, $s = \sin\alpha$

$d = \bar{u}_{x4} - \bar{u}_{x1} \Rightarrow F = \left[-\frac{H}{2s} + \frac{P c^2}{(1+2c^3)} \right]$ (tension & comp)

Same as before if $\alpha \to 0$, $F^{(1)}$ & $F^{(3)} \to$ "∞" because the structure cant sustain the horizontal force. Unstable system.

Assignment 2



$L^{(3)} = 5\sqrt{2}$, $E^{(3)}A^{(3)} = 200\sqrt{2}$

$L^{(1)} = 10$, $E^{(1)}A^{(1)} = 100$

$L^{(2)} = 10$, $E^{(2)}A^{(2)} = 50$

$L^{(4)} = 5\sqrt{2}$, $E^{(4)}A^{(4)} = 200\sqrt{2}$

Using the transformation of the stiffness matrix

$$K^{(1)}_{\phi=0} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(2)}_{\phi=\pi/2} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(3)} = \frac{200\sqrt{2}}{2\cdot 5\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$\phi = \pi/4$

$$K^{(4)} = 20 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$\phi = \frac{5\pi}{4}$

Expanded element stiffness:

Element 1:

$$K^{(1)} = 10 \left[\begin{array}{cc} \begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{array}\right]$$

Element 2:

$$K^{(2)} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3:

$$K^{(3)} = 20 \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & \Large 0 & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$K^{(4)} = 20 \begin{bmatrix} \Large 0 & \Large 0 \\ \Large 0 & \begin{matrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{matrix} \end{bmatrix}$$

Addet together to master stiffness matrix:

$$K = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & 0 & 0 & -5 & 20 & 25 & -20 & -20 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix}$$

Applying boundary conditions and outer forces we get the equlibrium:

$u_{x1} = u_{y1} = u_{y2} = 0$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Column 4 and 5 are identical

$\Rightarrow$ Linearly dependent matrix

$\Rightarrow$ Singular matrix $\Rightarrow$ solution blows up.

This means we cannot find the displacements when we know the forces

$\Rightarrow$ Cannot find the reactions either.

<u>Physically the structure is unstable</u>