International Centre for
Numerical Methods in Engineering

Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

Assignment 1

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1 ASSIGNMENT A): TRUSS STRUCTURE

PROBLEM DEFINITION

On “The Direct Stiffness Method”:

Consider the truss problem defined in the figure 1.1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

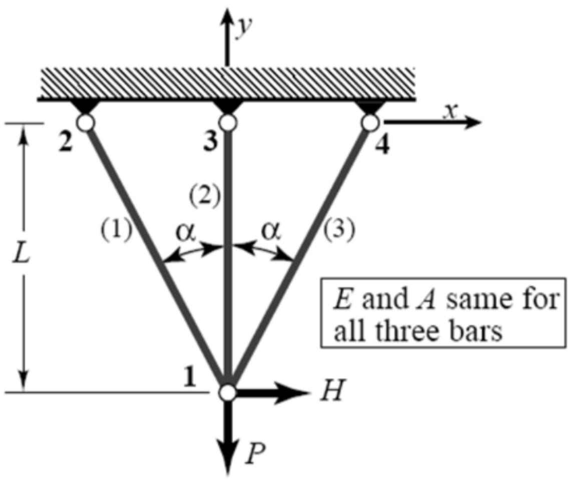

Figure 1.1: Truss structure. Geometry and mechanical features

1.1 OBTAIN THE MASTER STIFFNESS EQUATIONS.

The Direct Stiffness Method (DSM) is based in systematic steps organized in two main phases. The *Breakdown* phase, in which can be decomposed as:

a) Disconnection.

b) Localization.

c) Member Element Formation.

and *Assembly and Solution* phase which is formed by:

d) Globalization.

e) Merge.

f) Application of BC's.

g) Solution.

h) Recovery of derived quantities.

In order to obtain the master stiffness equations, first it is necessary to focus in the *Breakdown* phase in which the a) step is done by considering the figure 1.1 and separating each bar.

As the problem definition states, the structure is formed by three bars that can be considered as "*Truss Elements*" by following the Mechanics of Materials formulation. Now, the next steps are referred to b) localization and c) member element formation, that can be done by using the element stiffness matrix in local coordinates and considering the equivalent spring stiffness by the Young's Modulus "E", the area and the length of each element.

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} \tag{1.1}$$

Now, the next phase begins with the globalization of the truss formulation and to do that it is necessary to transform the local matrix 1.1 to a new one in a global system of coordinates employing a rotation matrix defined by the angle of the bar. To obtain this global matrix it is needed to pre-multiplicate the transpose of the rotation matrix with the stiffness matrix and then post-multiplicate it again:

$$K^e = (T^e)^T \bar{K} T^e \tag{1.2}$$

with:

$$T = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \tag{1.3}$$

Now, the global matrix of the truss element is:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \tag{1.4}$$

Now, particularizing the elemental global matrix for each bar of the problem we have:

- Bar 1
  - Nodes: 1 - 2
  - $\theta = -(90-\alpha)$
  - $\cos(\theta) = \sin(\alpha) = -s$
  - $\sin(\theta) = \cos(\alpha) = c$
  - Area: A
  - Young Modulus: E
  - Length: $L/\cos\alpha$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EAc}{L} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix} \tag{1.5}$$

- Bar 2
  - Nodes: 1 - 3
  - $\cos(\alpha) = \cos(90) = 0$
  - $\sin(\alpha) = \sin(90) = 1$
  - Area: A
  - Young Modulus: E
  - Length: L

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{1.6}$$

- Bar 3
  - Nodes: 1 - 4
  - $\theta = (90-\alpha)$
  - $\cos(\theta) = \sin(\alpha) = s$
  - $\sin(\theta) = \cos(\alpha) = c$
  - Area: A
  - Young Modulus: E
  - Length: $L/\cos\alpha$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \frac{EAc}{L} \begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{1.7}$$

Expanding the three matrices to the total of degrees of freedom and then adding them, we obtain the global stiffness matrix which corresponds to the e) merge step. The result is:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & SYM & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{1.8}$$

The matrix 1.8 corresponds exactly with the given by the problem. Now, to answer why the 5th row and column contain only zeros, first we have to take into account the numbering given to the nodes of the structure. The 5th row and column corresponds to the 5th degree of freedom of the whole structure which is the horizontal displacement of the 3rd node. As we can observe in the figure 1.1, the node 3 is not connected to any bar, so topologically reviewing there is no connection of that degree of freedom with another one, that results in zero elements added to the global matrix (in the 5th row and column because of the symmetry of said global matrix). Physically analyzing, the node 3 is the direct support of the bar 2, which is completely vertical, and as the truss matrix is implemented, this type of element just can handle axial forces, that means that the elemental stiffness matrix should have zeros in horizontal stiffness components, and do not can contribute to the global stiffness, as can be observed.

1.2 Apply the BCs and show the 2-equations modified stiffness system.

The boundary conditions can be Dirichlet type or Neumann type. The first ones corresponds to the imposed displacements (in solid mechanics), as it is observed in the figure 1.1, the node 2, 3 and 4 are constrained to zero displacement (horizontal and vertical), in other words, the rows and columns corresponding to these equations in the global system can be deleted in order to solve the problem. The Neumann BCs are related to the forces or fluxes in the system, so in the problem the only node that has this BC is the node 1, which have a vertical force P and an horizontal force H. So the system results in:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{1.9}$$

1.3 Solve for the displacements u_{x1} and u_{y1}.

The solution of the system of equations 1.9 is:

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} HL/2EAcs^2 \\ -PL/EA(2c^3+1) \end{bmatrix} \tag{1.10}$$

As it can be observed in the solution 1.10, both displacements are physically possible between the two boundary values $\alpha \to 0$ and $\alpha \to \pi/2$. The first one corresponds at the time the two inclined bars are getting closer to the vertical one. The problem in this limit case is such that as the three bars are becoming parallel, and the support is not constrained to rotate, the horizontal force is causing a huge displacement. The second case is when the bars are perpendicular, in that case the problem it will not clearly proposed, because if the vertical bar is still of length equal to L, the bars 1 and 3 would have a length equal to $L/\cos(\pi/2) = \infty$ which does not have a coherent physical solution. The next plots helps to understand the horizontal and vertical displacements in node 1 due the variation of the angle α.

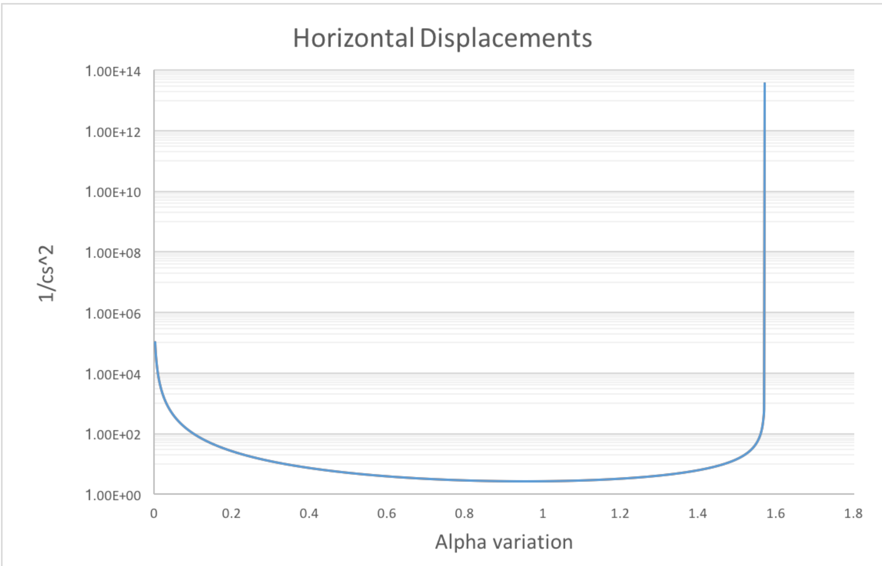

Figure 1.2: Horizontal displacement of node 1 due a variation of the angle α from 0 to $\pi/2$.

As explained above, the horizontal displacement u_{x1} "blows up" when $\alpha \to 0$ because there would be no equilibrium in the system as the structure will behave as a pendulus with no restriction with the rotation on the ceiling support and with a horizontal force pushing the bar in the node 1.

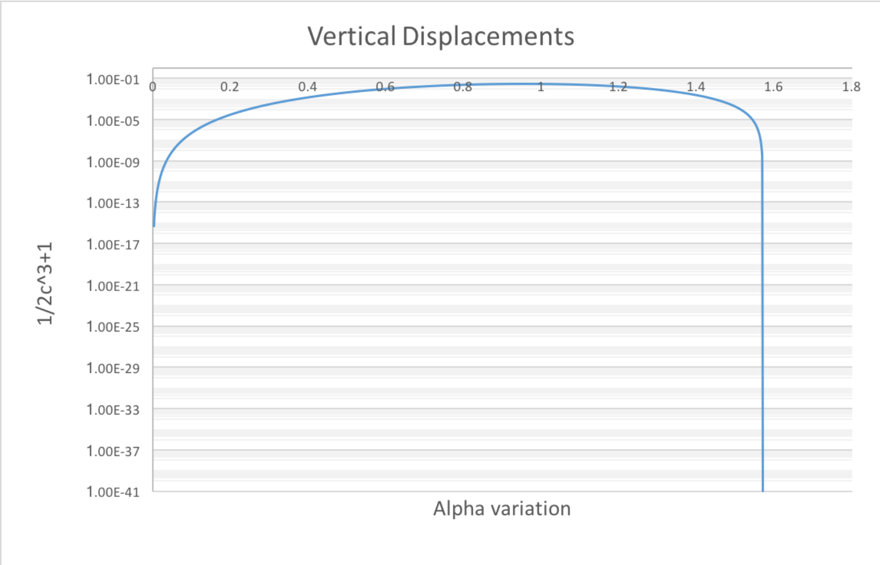

Figure 1.3: Vertical displacement of node 1 due a variation of the angle α from 0 to $\pi/2$.

1.4 Recover the axial forces in the three members.

To obtain the forces in each bar, first it can be obtained the displacements in the local system of coordinates in each bar. In that sense, first it is needed to multiply the rotation matrix T by the displacement vector:

- Bar 1

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} HL/2EAcs^2 \\ -PL/EA(2c^3+1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} HL/2EAcs + PLc/EA(2c^3+1) \\ HL/2EAs^2 - PLs/EA(2c^3+1) \\ 0 \\ 0 \end{bmatrix}$$

Then, the elongation in the bar 1 is defined as:

$$d^{(1)} = \bar{u}_{x1} - \bar{u}_{x2} = HL/2EAcs + PLc/EA(2c^3+1) - 0 = \frac{HL}{2EAcs} + \frac{PLc}{EA(2c^3+1)}$$

The axial force is given by the next expression:

$$F^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}} d^{(e)} \tag{1.11}$$

So, using this equation, the axial force in bar 1 is:

$$F^{(1)} = \frac{EAc}{L} d^{(1)} = \frac{H}{2s} + \frac{Pc^2}{2c^3+1}$$

- Bar 2

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} HL/2EAcs^2 \\ -PL/EA(2c^3+1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -PL/EA(2c^3+1) \\ -HL/2EAcs^2 \\ 0 \\ 0 \end{bmatrix}$$

Then, the elongation in the bar 2 is defined as:

$$d^{(1)} = \bar{u}_{y3} - \bar{u}_{y1} = 0 - (-PL/EA(2c^3+1)) = \frac{PL}{EA(2c^3+1)}$$

Now, using the equation 1.11, the axial force in bar 2 is:

$$F^{(2)} = \frac{EA}{L} d^{(2)} = \frac{P}{2c^3+1}$$

- Bar 3

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x4} \\ \bar{u}_{y4} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} HL/2EAcs^2 \\ -PL/EA(2c^3+1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} HL/2EAcs - PLc/EA(2c^3+1) \\ -HL/2EAs^2 - PLs/EA(2c^3+1) \\ 0 \\ 0 \end{bmatrix}$$

Then, the elongation in the bar 3 is defined as:

$$d^{(3)} = \bar{u}_{x4} - \bar{u}_{x1} = 0 - (HL/2EAcs - PLc/EA(2c^3+1) = \frac{-HL}{2EAcs} + \frac{PLc}{EA(2c^3+1)}$$

Computing the axial force for the bar 3:

$$F^{(3)} = \frac{EAc}{L} d^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{2c^3+1}$$

The reason why $F^{(1)}$ and $F^{(3)}$ “blows up” if $\alpha \to 0$ and $H \neq 0$ is similar than the explained above with the displacements. If α approximates to zero, the bars will be parallel to each other, and because of the supports which can rotate, the perpendicular force H will cause a system without equilibrium, i.e. the system will start to rotate.

2 Assignment b): Three Bars With Four Nodes

Consider the example of a truss with three bars and three nodes as in the 1st lesson. But, Dr. Who's proposed a "better" solution by adding a node as the figure 2.1 shows. To prove that this suggestion does not works, the usual procedure is used to compute the displacements.

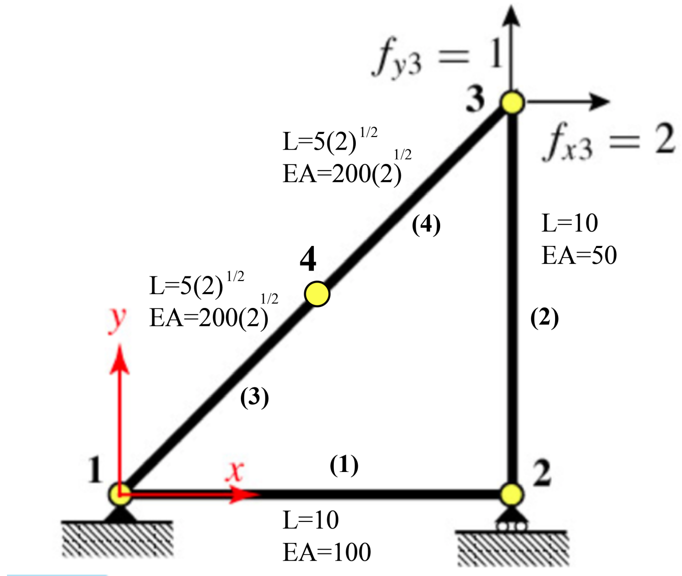

Figure 2.1: 3 Bar truss structure with 4 nodes. Geometry and mechanical features

First, computing the elemental matrices by using the equation 1.4:

- Bar 1
  - Nodes: 1 - 2
  - $\theta = 0$
  - $\cos(\theta) = 1$
  - $\sin(\theta) = 0$
  - EA=100
  - Length=10

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

- Bar 2
  - Nodes: 2 - 3
  - $\theta = 90$
  - $\cos(\theta) = 0$
  - $\sin(\theta) = 1$
  - EA=50
  - Length=10

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

- Bar 3
  - Nodes: 1 - 4
  - $\theta = 45$
  - $\cos(\theta) = \sqrt{2}/2$
  - $\sin(\theta) = \sqrt{2}/2$
  - $EA = 200\sqrt{2}$
  - $Length = 5\sqrt{2}$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x4} \\ f_{y4} \end{bmatrix} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

- Bar 4
  - Nodes: 4 - 3
  - $\theta = 45$
  - $\cos(\theta) = \sqrt{2}/2$
  - $\sin(\theta) = \sqrt{2}/2$
  - $EA = 200\sqrt{2}$
  - $Length = 5\sqrt{2}$

$$\begin{bmatrix} f_{x4} \\ f_{y4} \\ f_{x3} \\ f_{y3} \end{bmatrix} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Assembling the global stiffness matrix:

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & SYM & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{2.1}$$

Now, to determine if this system of equations can be solved, we can compute the determinant and verify if it is equal to zero, if this is true the matrix is singular and can not be inverted. Now, another test can be done by doing gaussian elimination, and observing that the last rows and columns have the same values, implementing this test will result in deleting a complete equation. For that reason this matrix is singular.

Physically, this means that adding a middle node in a bar will give two degrees of freedom on X and Y, moreover this node is not restricted to rotate. So, the structure will not be in equilibrium at the time the 4th node is collocated. The only possible way to permit implementing this suggestion is adding to this node proper boundary conditions to maintain the equilibrium or by using a modified elemental matrix with no rotation allowed.